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Almashtirish operatorlarining ta’rifi va tatbiqlari

Sh.T.Karimov
FarDU, amaliy matematika va informatika kafedrasi professori, fizika matematika fanlari doktori.
M.K. Xalilov
70540101 - Matematika (yo'nalishlar bo'yicha) mutaxassisligi magistranti

To‘plamlar nazariyasida to‘plamlar orasidagi munosabat asosan akslantirish orqali aniqlanadi. Biror X to‘plamni (yoki biror $E \subset X$ qismini) ikkinchi Y to‘plamga akslantirish uchun X ning har bir elementini Y to‘plamning biror tayin elementiga mos keltirish kerak. Xususiy holda X ni X ga (ya’ni to‘plamni o‘zini o‘ziga) akslantirish ham mumkin. To‘plamlar orasidagi aks ettirish jarayonini shartli ravishda $y = Ax$ $\left(x \in X, y \in Y\right)$ ko‘rinishida yozish mumkin.

1-ta’rif. *Agar* X *va* Y *ixtiyoriy to‘plamlar bo‘lsa (ya’ni sonli bo‘lmasligi ham mumkin), akslantirish qoidasi* A *operator deb yuritiladi.*

Demak, agar X to‘lamning har bir x elementiga tayin A qoida asosida Y to‘plamning bittagina y elementi mos keltirilgan bo‘lsa, X to‘plamda A operator berilgan deyiladi. X to‘plam A operatorning aniqlanish sohasi, x esa A operatorning argumenti deyiladi. $y = Ax$ ko‘rinishida tasvirlangan har bir $y \in Y$ element A operatorning qiymati yoki $x \in X$ elementning Y dagi tasviri deb ataladi.

Shunga e’tibor berish kerakki, $y = Ax$ tenglik ma’noga ega bo‘lishi uchun undagi A simvol X to‘plamni Y to‘plamga aks ettirish qoidalariga taaluqli barcha informatsiyalarni qabul qila olishi kerak.

Ma’lumki, differensial tenglamalar funksional to‘plamlarni bir-biriga aks ettiradi. Masalan,

$$\frac{d^2x(t)}{dt^2}+2p(t)\frac{dx(t)}{dt}+q(t)x(t)=y(t) \qquad (1.1)$$

differensial tenglamada $p(t)$, $q(t)$ uzluksiz funksiyalar, $x(t)$ ni esa ikkinchi tartibli hosilagacha uzluksiz funksiya deb faraz qilinsa, u holda (1) differensial tenglama ikkinchi tartibli hosilagacha uzluksiz bo‘lgan funksiyalar sinfi $C^{(2)}$ ni

uzluksiz funksiyalar sinfi C ga aks ettiradi. Operatorni

$$A=\frac{d^2}{dt^2}+2p\frac{d}{dt}+q \qquad (1.2)$$

ifoda yordamida kiritsak, (1) ushbu

$$Ax=y$$

ko‘rinishga keltiriladi.

Bunday misollarni ko‘plab keltirish mumkin. Faqat hamma hollarda operatorli tenglikdan asliga o‘tish qoidasi, ya’ni operatorning ma’nosi formulalar yoki ta’riflar orqali aniq berilishi kerak.

Demak, $y=Ax$ operator berilishi uchun: a) ikkita X va Y; b) A operatorning aniq ma’nosi berilishi kerak.

Yana shuni ta’kidlash kerakki, operator orqali uchraydigan deyarli hamma tenglamalarni yagona usul bilan ifodalash mumkin, bu har xil masalalarni umumiy nuqtayi nazardan qarab, ularni tekshirish imkonini beradi.

Operator tenglamalarni o‘rganilayotganda eng avval shu operatorlarning aniqlanish sohasi va qiymatlar sohasiga e’tibor qilinadi. Shu to‘plamlarning berilishiga qarab operatorli tenglamalarning yechish usuli ham har xil bo‘ladi.

2-ta’rif. (A,B) *operatorlar jufti berilgan bo‘lsin. Agar*

$$TA=BT \qquad (1.3)$$

munosabat bajarilsa, noldan farqli T *operator almashtirish operatori (AO-transmutation) deb ataladi.*

(3) munosabat boshqacha aytganda, o‘ram xususiyatga ega bo‘ladi, u holda AO A va B operatorlarni o‘raydi. (3) munosabatning qat’iy ta’rifga aylanishi uchun A va B operatorlar aniqlangan fazolarni yoki funksiyalar to‘plamlarini berish zarur. Ba’zan AO ta’rifiga teskari operatorni mavjud bo‘lish sharti ham keltiriladi, bu esa zarur hususiyat bo‘ladi. Aniq ishlarda A va B operatorlar odatda differensial bo‘ladi, T - standart fazolardagi chiziqli operator. Aniqki, AO tushunchasi algebradagi matritsalar o‘xshashligi tushunchasining umumlashganidir va chiziqlidir. Ammo AO lar o‘xshash (yoki ekvivalent) operatorlarga keltirilmaydi, chunki o‘ram operatorlar qoidaga ko‘ra tabiiy fazolarda chegaralanmagan bo‘ladi. AO ga teskari operator xuddi o‘sha fazoda mavjud va aniqlangan bo‘lishi shart emas. AO ning bog‘lam operatorlar spektorlari mos tushmasligi mumkin. Odatda almashtirish operatorlari qanday qo‘llaniladi? Masalan, biz qandaydir yetarlicha murakkab tuzilgan A operatorni o‘rganaylik. Bunda kerakli xususiyatlar modelli yana ham soda B operator uchun ma’lumdir. U holda, AO (1.3) mavjud bo‘lsa, ko‘pincha B operator xususiyatlarini A ga ko‘chirishga erishiladi. AO ni aniq masalalarda qo‘llashga keltirilgan sxema ana shunday bo‘ladi.

AO nazriyasining berilishi va tadbiqlariga [1-17] monografiyalarning haqiqiy qismlari bag‘ishlangan. Bitta terminologik izoh qilamiz. G‘arb adabiyotlarida AO uchun “transmutation” degan atama qabul qilingan. Bu atama J.Delsart ismi bilan bog‘liqdir. R.Kerol ko‘rsatishicha shunga o‘xshash “transformation” degan atama bunda Fure, Laplas, Mellinlarning va shunga o‘xshash boshqa klassik integral almashtirishlarga biriktiriladi. Bundan tashqari, “transmutation” degan atama rumin tillarida qo‘shimcha nom “sehrli aylanish” nomini olgan. Bu AO ma’nosini aniq bildiradi.

AO nazariyasining zarurligi uning ko‘p sonli tatbiqlari bilan isbotlangan AO usullari Furening umumlshgan almashtirishlarini, spektral funksiyani va Gelfanda-Levitanning mashhur tenglamasining yechimini aniqlagan holda teskari masalalar nazariyasida keng qo‘llaniladi; AO orqali to’lqin tarqalish nazriyasida Marchenkoning ham mashhur tenglamasi olinadi; spektral nazariyada izlarning ma’lum bo‘lgan formulalari va spektral funksiya asimptotikasi olinadi; chiziqli bo‘lmagan differensial tenglamalar nazariyasida Laks usuli AO ni yechimlar mavjudligini isbotlash va solitonlarni qurishga qo‘llaniladi. AO ning aniq turlari bu- umumlashgan analitik funksiyalar nazariyasining va umumlashgan siljish operatorlarining qismlaridan iborat. Xususiy hosilali tenglamalar nazariyasida AO usullari ba'zi masalalarning aniq yechimlarini qurish, singulyar va buziladigan differensial tenglamalar uchun chegaraviy masalalarni, psevdodifferensial operatorlarni, chegara qismida ajoyib yechimli masalalarni o'rganish, ba'zi elliptik va ultra elliptik tenglamalar yechimlari kamayish tezligini baholashni o'rganishda qo'llaniladi. AO nazariyasi xos funksiyalar va yadrodagi xos funksiyali integral operatorlar qandaydir yangi klassifikatsiyasini berish imkonini yaratadi, shuningdek, kasr integrodifferensiallashning turli xil operatorlarini tuzish imkonini beradi. Funksiyalar nazariyasida AO ning funksional fazolar va Xardi operatorlarini umumlashtirish, Peli–Viner nazariyasini kengaytirishga tadbiqlari topilgan. AO usullari katta muvoffaqqiyat bilan ko'pgina tatbiqiy masalalarda qo'llanilmoqda: masalan, kvant tarqalish nazariyasida, ehtimollar nazariyasi va tasodifiy jarayonlarda, chiziqli stoxastik baholashda, filtratsiyada, stoxastik tasodifiy tenglamalarda, geofizika va transtovushli gazodinamikaning teskari masalalarida.

Ikkinchi tartibli hosilali Bessel differensial operatorini o'raydigan, masalan, eng mashhur AO lar sinfini ko'rib chiqishga o'tamiz:

$$T(B_\nu)f=(D^2)Tf\,,\;\; B_\nu=D^2+\frac{2\nu+1}{x}D,\;\;\; D^2=\frac{d^2}{dx^2},\quad \nu\in C\,. \tag{1.4}$$

AO ni qurish usullaridan biri bu- mos keluvchi differensial tenglamalar yechimlari orasidagi mosliklarni qo'yishdir. $B_\nu f=\lambda f$ ko'rinishdagi tenglamalar yechimlari Bessel funksiyalaridir, $D^2 f=\lambda f$ ko'rinishdagi tenglamalar esa trigonometrik funksiyalar yoki eksponenta. Shuning uchun, (4) ko'rinishdagi AO larning asli bo'lib Puasson va Sonin formulalari xizmat qiladi:

$$J_\nu(x)=\frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)2^{\nu-1}x^\nu}\int_0^x\left(x^2-t^2\right)^{\nu-\frac{1}{2}}\cos(t)dt\,,\ \mathrm{Re}\nu>\frac{1}{2} \tag{1.5}$$

$$J_\nu(x)=\frac{2^{\nu+1}x^\nu}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\int_x^\infty\left(t^2-x^2\right)^{-\nu-\frac{1}{2}}\sin(t)dt\,,\ -\frac{1}{2}<\mathrm{Re}\nu<\frac{1}{2} \tag{1.6}$$

Odatda, formulalar atama nomlari shartli xarakterga ega. Ixtiyoriy indeks uchun Bessel funksiyalari buyuk Leonard Eyler tomonidan kiritilgan. Umuman, bu funksiyalar Rikati nomi bilan ataladigan differensial tenglamani yechish uchun paydo bo‘lgan va sekin asta 1690-1770-yillari oralig‘ida kiritilgan. Bunda Bernuli oilasi a’zolari, Venesiyalik graf Jakkolo Franchesko Rikatti va Eylerlar qatnashgan. Shu bilan birga, Fridrix Besselning o‘zining nomi bilan ataluvchi funksiyalarni o‘ganishga qo‘shgan xissasi beqiyos ulkandir. (5) integralni Eyler 1769-yida o‘rganishni boshlagan. So‘ngra Parseval integralini $\nu=0$ da 1805-yilda hisobladi; butun ν lar uchun (5) formulani Plana 1821-yilda olgan; Puasson uni 1823-yilda yarim butun bo‘lgan ν lar uchun keltirib chiqardi. Uning usuli butun ν lar uchun ham qo‘llaniladi, lekin buni o‘zi sezmadi. Keyinchalik, ushbu integral Kummer, Lobbato va Dyuamel ishlarida uchradi. Umuman, biz Puassonga yopishtirayotgan (5) formulani keltirib chiqargan.

3-ta’rif. *Puassonning AO i deb quyidagi ifodaga ataladi:*

$$P_\nu f=\frac{1}{\Gamma(\nu+1)2^\nu x^\nu}\int_0^x\left(x^2-t^2\right)^{\nu-\frac{1}{2}}f(t)dt,\ \operatorname{Re}\nu>-\frac{1}{2} \tag{1.7}$$

Sonin AO i deb quyidagi ifodaga aytiladi:

$$S_\nu f=\frac{2^{\nu+\frac{1}{2}}x^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\frac{d}{dx}\int_0^x\left(x^2-t^2\right)^{-\nu-\frac{1}{2}}t^{2\nu+1}f(t)dt,\ \operatorname{Re}\nu<\frac{1}{2}. \tag{1.8}$$

(7)-(8) opertorlar AO kabi

$$S_\nu B_\nu=D^2S_\nu,\ \ P_\nu D^2=B_\nu P_\nu \tag{1.9}$$

formulalar bo‘yicha xizmat qiladi. Ularni $\nu\in C$ *ning hamma qiymatlari uchun aniqlash mumkin.*

(7)-(8) operatorlarga o‘xshash operatorlarni o‘rganish g‘oyasini Liuvil aytib o‘tgan edi, ularni Besselning funksiyalari nazariyasi konteksida real qo‘llash Nikolay Yakovlevich Sonin boshlagan. AO kabi bu opetratorlar Delsart ishlarida [18,19] kiritilgan va so‘ngra Delsart va Lions ishlarida [19-22] o‘rganilgan. Shuning uchun, biz (7)-(8) larni Sonin-Puasson-Delsart (SPD) AO i deb ataymiz. Bizning mamlakatimizda SPD operatorlari haqida asosan B.M.Levitanning [23] ajoyib yozilgan maqolasidan bilib olingan.

SPD (7)-(8) operatorlari butun AO nazariyasining eng mashhur ob'ektlari desak, mubolag'a bo'lmaydi, ularni o'rganish, tatbiq qilish va umumlashtirishga yuzlab ishlar bag'ishlangan. Qisqacha faqat asosan yo'nalishlarni sanab o'tamiz.

Delsart tomonidan AO SPD bazasida umumlashgan siljishning fundamental tushunchasi keltirilgan.

4-ta'rif. Umumlashgan siljish operatori (USO) deb,

$$(B_\nu)_y u(x,y) = \left(\frac{\partial^2}{\partial y^2} + \frac{2\nu+1}{y}\frac{\partial}{\partial y} \right) u(x,y) = \frac{\partial^2}{\partial x^2} u(x,y),$$

$$u(x,0) = f(x),\ u_y(x,0) = 0 \qquad (1.10)$$

masalaning $u(x,y) = T_x^y f(x)$ yechimiga aytiladi. Ta'rif shu bilan tushuntiriladiki, xususiy holda $\nu = -1/2$ (USO) $T_x^y f(x) = \frac{1}{2}(f(x+y) + f(x-y))$ oddiy siljishiga keltiriladi. (4) (USO) uchun Delsart tomonidan aniq

$$T_x^y f(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos(t)}\right) \sin^{2\nu} t dt \qquad (1.11)$$

formula olingan. (10) ta'rifda differensial operatorlarning ixtiyoriy juftliklarini ham ko'rib chiqish mumkin. Masalan, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$, $u(x,0) = f(x)$ ta'rifda odatda $T_x^y f(x) = f(x+y)$ siljishga ega bo'lamiz.

Endi, yangi siljish borligi uchun, odatdagi siljishga asoslangan avvalo nazariyalarni umumlashtirish mumkin bo'ladi. Shunday qilib, umumlashgan davriy bo'lgan funksiyalar nazariyasi, (shuni belgilaymizki, davriy bo'lgan funksiyalarning ta'rifi va asosiy xossalari birinchi marta Yurevskiy (Tartusskiy (Derpskiy)) universiteti professori Pirs Georgievich Bol tomonidan, T. Borgacha berilgan edi, shuningdek, Bol Brauergacha sferaning o'ziga uzluksiz akslangan siljimaydigan nuqtalari haqidagi mashhur topologik teoremani ham hosil qiladi; (revolyusiya va urush vaqtida evakuatsiya qilingan Yuevskiy universitet xodimlaridan 1918 - yilda Voronej universiteti tuzila boshlagan edi,) haqiqatda, Teylar-Delsart qarorlari deb atalgan Teylorning umumlashgan qatorlari bo‘yicha taqsimlanish, umumlashgan aylanish va uning tatbiqlari kelib chiqqan.

Xususiy hosilali tenglamalar nazariyasida AO va USO konstruksiyalari katta ro‘l o‘ynaydi. AO lar ancha murakkab tenglamalarni anchagina soddalariga aylantirishga imkon beradi, USO lar esa singulyar tenglamalarda asosiylikni (maxsuslikni) boshidan ixtiyoriy nuqtaga ko‘chirishdaga yordam beradi, shuningdek, umumlashgan aylanma yordamida fundamental yechimlarni topishga yordam beradi.

$$\sum_{k=1}^{n} B_{\nu,x_k} u(x_1,...,x_n) = f \qquad (1.12)$$

ko‘rinishdagi har bir o‘zgaruvchi bo‘yicha Bessel operatorlari bilan B-elliptik tenglama bo‘lgan xususiy hosilali tenglamalarning bitta sinfini alohida belgilab o‘tamiz, B-giperbolik va B-parabolik tenglamalar shunga o‘xshash ko‘rib chiqiladi. Ushbu sinf tenglamalarini o‘rganish Eyler, Puasson, Darbu ishlarida boshlangan edi, A.Vaynshteynning umumlashgan o‘qsimmetrik potensiali nazariyasida davom ettirildi va I.Y.Egorov, YA.I.Jitomirskiy, L.D.Kudrevsev, P.I.Lizorkin, M.I.Matiychuka, L.G.Mixaylov, M.N.Olevskiy, M.M.Simirnov, S.A.Tersenova, Xe Kan Chera, A.I.YAnushauskas va boshqa mahalliy matematiklar uchun hamma masalalar yanada to‘lig‘icha Voronej matematigi Ivan Aleksandrovich Kipriyanov va uning o‘quvchilari tomonidan o‘rganib chiqilgan edi; Mos tenglamalar yechimlarining sinfini aniqlash uchun I.A.Kipriyanov tomonidan [24] funksional fazolar kiritilgan va o‘rganilgan, keyinchalik, uning nomi bilan atalgan.

Ushbu yo‘nalishda V.V.Katraxov ish olib borgan. Hozir Bessel operatorli tenglamalar va ular bilan bog‘liq masalalarni A.V.Glushak, V.S.Guliev, L.N.Lexov o‘zlarining hamkasblari va shogirtlari bilan o‘rganishmoqda. [25] mashhur monografiyada boshlanadigan (12) ko‘rinishdagi operatorli tenglamalar uchun masalalarni A.V.Glushak, S.B.Shmulevich, V.D.Fernikov va boshqalar ko‘rib chiqishgan.

Bu yerda shuni takidlash kerakki, o‘zgaruvchi koeffitsentli xususiy hosilali singulyar va buziladigan differensial tenglamalarni o‘rganish hisobi boshlangan birinchi fundamental ish bu M.V.Keldishning [26] maqolasidir. Elliptik tipdagi buziladigan tanglamalarning umumiy nazariyasi keyinchalik G.Fikera, O.A.Oleynik, E.A.Radkevich, V.N.Vrachov, V.P.Glushko va boshqalar tomonidan qayta ishlab chiqilgan.

AO SPD larni o‘rganish bo‘yicha qiziqarli natijalar V.V.Katraxov tomonidan olingan, ular shuningdek R.Keroll tomonidan maxsus qayta ishlangan ko‘rinishda [7] ning alohida bo‘limida berilgan. AO lar shuningdek, Bessel operatorlarining ko‘p sonli umumlshgan ko‘rinishlari uchun ham ko‘rib chiqilgan .

(10) ko‘rinishdagi A operatorlarning nazariya uchun muhimligi shundaki, ular Gelfandning mashhur formulasi bo‘yicha simmetrik fazolarda Laplas operatorlarining radial qismini ko‘rsatadi [16]. Bunda Bessel operatori (10) da $\nu(x)=x^{2\nu+1}$ ni tanlashda hosil bo‘ladi. AO orqali yondashish yagona holatlarda (13) xos funksiyalar ifodalangan maxsus funksiyalarining ko‘p xossalarini o‘rganish imkonini beradi.

AO lar qurilgan boshqa model operator- bu Eyri D^2+x operatoridir. Kvant mexanikasidagi Shtark effekti bilan boshlangan uning buziladigan varianti ham ko‘rib chiqilgan.

Sonin-Puasson-Delsart operatorlarining muhim umumlashgani bu giperbessel funksiyalari uchun AO lardir, ular Ivan Dimovskix va uning shogirtlari ishlarida har tomonlama o‘rganilgan. Mos keluvchi AO lar adabiyotlarda Sonin-Dimovski va Puasson-Dimovski nomlariga xizmati bilan ega bo‘lgan, ular shuningdek, Ivan Dimovskining shogirti Virjiniya Kiryakova ishlarida ham o‘rganilgan.

SPD operatorlarining boshqa muhim umumlashgan sinfi Dunkl operatorlarini o‘rganishda paydo bo‘ldi. Dunkl operatorlari AO larning umumiy sxemasiga maxsus hol sifatida kiritiladi, o‘raladigan operatorlarning biri differensial turlicha, ikkinchisi esa avvalgidek, differensial bo‘ladi. Dunkl operatorlari uchun Sonin va Puasson tipidagi AO larni qo‘shishga oxirgi vaqtlarda juda ko‘p ishlar bag‘ishlangan.

Xossalar bilan butun ν larning xususiy xoli uchun mos tenglamalarni yechishdagi AO ni qurishning qandaydir usuli Kram-Kreyn uslubi nomi asosida adabiyotlarda eslatiladi, ammo u murakkablashtirilgan chegirmalarning uzun ketma-ketligiga olib keladi, natijaviy AO esa aniq ko‘rinishda chiqarila olmaydi. Haqiqatdan, Kram-Kreyn chegirmalari ham AO larning umumiy sxemasiga kiritilishi mumkin. Ular o‘raluvchi operatorlar differensial bo‘lib qolgan holda paydo bo‘ladi, AO esa ushbu juftlik uchun integral tasvirda emas, balki differensial tasvirda izlanadi. Yanada umumiy ko‘rinishda bu nazariya Darbu [Matveev] almashtirish usuli sifatida ma’lumdir. Xususiy xoldagi Kram-Kreyn chegirmalari shularga kiradi. Bunday differensial AO larga Muara almashtirishlari tipidagi ko‘p ma’lum bo‘lgan o‘zgaruvchilarning almashishi ham tegishli bo‘lishi mumkin. Ular linearizatsiya (chiziqlashtirish) masalasini yechgan holda chiziqli bo‘lmagan va chiziqli bo‘lgan operatorlarni o‘raydi.

Ikki va yuqori tartibli B-giperbolik va B-parabolik tenglamalar uchun [27-41] ishlarda AO sifatida Erdeyi-Kober kasr tartibli operator olingan va u yordamida turli xil masalalar tadqiq etilgan.

Almashtirish operatori

Shturm-Liuvill operatorlari spektral nazariyasining teskari masalaJarini o'rganishda almashtirish operatorlari muhim rol o‘ynaydi. Ular ikkita har xil Shturm-Liuvill tenglamasining yechimlarini o‘zaro bog‘laydi. Almashtirish operatorlari ilk bor B.M.Levitan va J.Delsartlarning ilmiy ishlarida vujudga kelgan. Bu operator ixtiyoriy Shturm-Liuvill tenglamasi uchun A.Povzner tomonidan qurilgan. Spektral analizning teskari masalasini yechishda almashtirish operatorlari I.M.Gelfand, B.M.Levitan va V.A.Marchenkolar tomonidan foydalanilgan.

Quyidagi

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{2.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.2}$$

Shturm-Liuvill chegaraviy masalasini qaraylik. Bu yerda λ spektral parametr, $q(x) \in C[0,\pi]$ haqiqiy funksiya va $h,\ H$ chekli haqiqiy sonlar.

$c(x,\lambda),\ s(x,\lambda),\ \varphi(x,\lambda),\ \psi(x,\lambda)$ funksiyalar orqali (2.1) tenglamaning mos ravishda quyidagi

$c(0,\lambda)=1,\ \ c'(0,\lambda)=0;\ \ s(0,\lambda)=0,\ \ s'(0,\lambda)=1;$

$\varphi(0,\lambda)=1,\ \ \varphi'(0,\lambda)=h;\ \ \psi(\pi,\lambda)=1,\ \ \psi'(\pi,\lambda)=-H.$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz.

Teorema 2.1. *Shturm-Liuvill tenglamasining* $c(x,\lambda)$ *yechimi uchun ushbu*

$$c(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t\,dt \tag{2.3}$$

integral tasvir o‘rinli: Bu yerda $K(x,t)$ *haqiqiy uzluksiz funksiya bo‘lib,*

$$K(x,x)=\frac{1}{2}\int_0^x q(t)\,dt \tag{2.4}$$

shartni qanoatlantiradi.

Isbot. $c(x,\lambda)$ funksiya uchun ushbu

$$c(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x \frac{\sin\sqrt{\lambda}(x-\tau)}{\sqrt{\lambda}}q(\tau)c(\tau,\lambda)\,d\tau, \tag{2.5}$$

integral tenglama olingan edi. Bu integral tenglamada

$$\frac{\sin\sqrt{\lambda}(x-\tau)}{\sqrt{\lambda}}=\int_\tau^x \cos\sqrt{\lambda}(t-\tau)\,dt,$$

formuladan foydalansak. (2.5) tenglama quyidagi ko'rinishni oladi:

$$c(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x q(\tau)c(\tau,\lambda)\left(\int_\tau^x \cos\sqrt{\lambda}(t-\tau)\,dt\right)d\tau.$$

Integrallash tartibini almashtirib, oxirgi tenglamani

$$c(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x\left(\int_0^t q(\tau)c(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)\,d\tau\right)dt,$$

ko'rinishda yozib olamiz. Hosil bo'lgan ikkinchi tur Volterra integral tenglamasini ketma-ket yaqinlashish usulidan foydalanib yechamiz. Buning uchun $c(x,\lambda)$ yechimni

$$c(x,\lambda)=\sum_{n=0}^{\infty}c_n(x,\lambda), \tag{2.6}$$

qator ko'rinishida izlaymiz. Bu yerda

$$c_0(x,\lambda)=\cos\sqrt{\lambda}x$$

$$c_{n+1}(x,\lambda)=\int_0^x\left(\int_0^t q(\tau)c_n(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)\,d\tau\right)dt. \tag{2.7}$$

Endi, matematik induksiya usulidan foydalanib, $c_n(x,\lambda)$, $n=1,\ 2,\ \ldots$ funksiyalarni ushbu

$$c_n(x,\lambda)=\int_0^x K_n(x,t)\cos\sqrt{\lambda}t\,dt, \tag{2.8}$$

ko'rinishida tasvirlanishini ko'rsatamiz. Bu yerda $K_n(x,t)$ funksiya λ spectral parametrga bog'liq emas.

Dastavval $c_1(x,\lambda)$ funksiyani ushbu

$$\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

tenglikdan foydalanib hisoblaymiz:

$$c_1(x,\lambda)=\int_0^x\left(\int_0^l q(\tau)\cos\sqrt{\lambda}\tau\cos\sqrt{\lambda}(t-\tau)d\tau\right)dt=$$
$$=\frac{1}{2}\int_0^x\cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$
$$+\frac{1}{2}\int_0^x\left(\int_0^t q(\tau)\cos\sqrt{\lambda}(t-2\tau)d\tau\right)dt.$$

Ikkinchi integralda $t-2\tau=s$ almashtirish bajarib,

$$c_1(x,\lambda)=\frac{1}{2}\int_0^x\cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$
$$+\frac{1}{4}\int_0^x\left(\int_{-t}^t q\left(\frac{t-s}{2}\right)\cos\sqrt{\lambda}sds\right)dt,$$

bo'lishini topamiz. Bu tenglikning o'ng tomonidagi ikkinchi integralda integrallash tartibini almashtiramiz. Natijada quyidagi

$$c_1(x,\lambda)=\frac{1}{2}\int_0^x\cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$
$$+\frac{1}{4}\int_0^x\cos\sqrt{\lambda}s\left(\int_s^x q\left(\frac{t-s}{2}\right)dt\right)ds+$$
$$+\frac{1}{4}\int_{-x}^0\cos\sqrt{\lambda}s\left(\int_{-s}^x q\left(\frac{t-s}{2}\right)dt\right)ds=$$
$$=\frac{1}{2}\int_0^x\cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$
$$+\frac{1}{4}\int_0^x\cos\sqrt{\lambda}s\left(\int_s^x\left[q\left(\frac{t-s}{2}\right)+q\left(\frac{t+s}{2}\right)\right]dt\right)ds,$$

tenglikni hosil qila.miz. Shunday qilib, (2.8) tenglik $n=1$ holda to'g'riligiga ishonch hosil qildik:

$$c_1(x,\lambda)=\int_0^x K_1(x,t)\cos\sqrt{\lambda}tdt,$$
$$K_1(x,t)=\frac{1}{2}\int_0^t q(\tau)d\tau+\frac{1}{4}\int_t^x\left[q\left(\frac{s-t}{2}\right)+q\left(\frac{s+t}{2}\right)\right]ds=$$
$$=\frac{1}{2}\int_0^{\frac{x+t}{2}} q(\xi)d\xi+\frac{1}{2}\int_0^{\frac{x-t}{2}} q(\xi)d\xi,\quad t\le x. \qquad (2.9)$$

Faraz qilaylik (2.8) tenglik biror $n\ge1$ holda o'rinli bo'lsin.
U holda (2.8) tenglikni (2.7) formulaga qo'yib,

$$c_{n+1}(x,\lambda)=\int_0^x\int_0^t q(\tau)\cos\sqrt{\lambda}(t-\tau)\int_0^\tau K_n(\tau,s)\cos\sqrt{\lambda}s\,ds\,d\tau\,dt=$$

$$=\frac{1}{2}\int_0^x\int_0^t q(\tau)\int_0^\tau K_n(\tau,s)\left[\cos\sqrt{\lambda}(s+t-\tau)+\cos\sqrt{\lambda}(s-t+\tau)\right]ds\,d\tau\,dt,$$

tenglikni hosil qilamiz. Bu yerda avvalo integralni ikkiga ajratib so'ngra ushbu $s+t-\tau=\xi$ va $s-t+\tau=\xi$ almashtirishlarni mos ravishda bajarsak, quyidagi tenglikka kelamiz:

$$c_{n+1}(x,\lambda)=\frac{1}{2}\int_0^x\int_0^t q(\tau)\int_{t-\tau}^{t} K_n(\tau,\xi+\tau-t)\cos\sqrt{\lambda}\xi\,d\xi\,d\tau\,dt+$$

$$+\frac{1}{2}\int_0^x\int_0^t q(\tau)\int_{\tau-t}^{2\tau-t} K_n(\tau,\xi+t-\tau)\cos\sqrt{\lambda}\xi\,d\xi\,d\tau\,dt\,.$$

Oxirgi tenglikda integrallar tartibini o'zgartirib,

$$c_{n+1}(x,\lambda)=\int_0^x K_{n+1}(x,t)\cos\sqrt{\lambda}t\,dt,$$

ekanini topamiz. Bu yerda

$$K_{n+1}(x,t)=\frac{1}{2}\int_t^x\left[\int_{\xi-t}^{\xi} q(\tau)K_n(\tau,t+\tau-\xi)\,d\tau+\right.$$

$$\left.+\int_{\frac{\xi+t}{2}}^{\xi} q(\tau)K_n(\tau,t-\tau+\xi)\,d\tau+\int_{\frac{\xi-t}{2}}^{\xi-t} q(\tau)K_n(\tau,-t-\tau+\xi)\,d\tau\right]d\xi. \qquad (2.10)$$

Endi (2.8) tenglikni (2.6) formulaga qo'yib, (2.3) tenglikni hosil qilamiz. Bu yerda

$$K(x,t)=\sum_{n=1}^{\infty}K_n(x,t). \qquad (2.11)$$

(2.9) va (2.10) tengliklardan foydalanib, ushbu

$$|K_n(x,t)|\le(Q(x))^n\frac{x^{n-1}}{(n-1)!},\qquad Q(x)=\int_0^x|q(\xi)|\,d\xi, \qquad (2.12)$$

tengsizlikni hosil qilamiz.

Haqiqatan ham, (2.9) tenglikdan $t\le x$ bo'lganda

$$|K_1(x,t)|\le\frac{1}{2}\int_0^{\frac{x+t}{2}}|q(\xi)|\,d\xi+\frac{1}{2}\int_0^{\frac{x-t}{2}}|q(\xi)|\,d\xi\le\int_0^x|q(\xi)|\,d\xi=Q(x),$$

kelib chiqadi.

Agar biror $n\ge1$ uchun (2.12) tengsizlik bajarilsa, u holda (2.10) tenglikdan foydalanib,

$$\left|K_{n+1}(x,t)\right| \le \frac{1}{2}\int_t^x \left[\int_{\frac{\xi+t}{2}}^{\xi} |q(\tau)|\left(Q(\tau)\right)^n \frac{\tau^{n-1}}{(n-1)!}d\tau + \right.$$

$$\left. + \int_{\frac{\xi-t}{2}}^{\xi} |q(\tau)|\left(Q(\tau)\right)^n \frac{\tau^{n-1}}{(n-1)!}d\tau \right] d\xi \le$$

$$\le \int_0^x \int_0^{\xi} |q(\tau)|\left(Q(\tau)\right)^n \frac{\tau^{n-1}}{(n-1)!}d\tau d\xi \le$$

$$\le \int_0^x \left(Q(\xi)\right)^{n+1} \frac{\xi^{n-1}}{(n-1)!}d\xi \le \left(Q(x)\right)^{n+1}\frac{x^n}{n!},$$

baholashga ega bo'lamiz. (2.12) tengliklardan foydalanib, (2.11) funksional qatorning $0 \le t \le x \le \pi$ sohada absolyut va tekis yaqinlashishini payqash qiyinchilik tug'dirmaydi. (2.9)- (2.11) formulalardan $K(x,t)$ funksiyaning silliqlik darajasi, ushbu

$$\int_0^x q(t)dt$$

funksiyaning silliqlik darajasi bilan bir xil ekanligi kelib ehiqadi. (2.9) va (2.10) tengliklarga ko'ra

$$K_1(x,x) = \frac{1}{2}\int_0^x q(t)dt, \qquad K_{n+1}(x,x) = 0, \qquad n \ge 1$$

bo'lgani uchun

$$K(x,x) = \frac{1}{2}\int_0^x q(t)dt$$

bo'ladi.

Xuddi shuningdek $s(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlar uchun ham almashtirish operatorlarining ko'rinishini topish mumkin.

Teorema 2.2. *Shturm-Liuvill tenglamasining $s(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlari uchun*

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x P(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt, \qquad (2.13)$$

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x G(x,t)\cos\sqrt{\lambda}tdt, \qquad (2.14)$$

integral tasvirlar o'rinli. Bu yerda $P(x,t)$ va $G(x,t)$ haqiqiy uzluksiz funksiyalar bo'lib, ularning silliqligi ushbu

$$\int_0^x q(t)\,dt$$

funksiyaning silliqligi bilan bir xil bo'ladi va ushbu

$$G(x,x)=h+\frac{1}{2}\int_0^x q(t)\,dt, \tag{2.15}$$

$$P(x,x)=\frac{1}{2}\int_0^x q(t)\,dt, \tag{2.16}$$

tengliklar bajariladi.

Isbot. $s(x,\lambda)$ funksiya quyidagi

$$s(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}q(t)s(x,\lambda)\,dt,$$

integral tenglamani qanoatlantiradi. Bu integral tenglamani quyidagi ko'rinishda yozib olamiz**:**

$$s(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\int_0^t q(\tau)s(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)\,d\tau dt.$$

Oxirgi integral, tenglamaga ketma-ket. yaqinlashish usulini qo'llaymiz:

$$s(x,\lambda)=\sum_{n=0}^{\infty}s_n(x,\lambda) \tag{2.17}$$

$$s_0(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$$

$$s_{n+1}(x,\lambda)=\int_0^x\int_0^t q(\tau)s_n(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)d\tau dt.$$

Xuddi teorema 2.1 da qo'llanilgan usuldan foydalanib, ushbu

$$s_n(x,\lambda)=\int_0^x P_n(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt,\quad n\ge 1$$

tasvirni olamiz. Bu yerda

$$P_1(x,t)=\frac{1}{2}\int_0^{\frac{x+t}{2}} q(\xi)\,d\xi-\frac{1}{2}\int_0^{\frac{x-t}{2}} q(\xi)\,d\xi$$

$$P_{n+1}(x,t)=\frac{1}{2}\int_t^x\left[\int_{\xi-t}^{\xi} q(\tau)P_n(\tau,t+\tau-\xi)\,d\tau+\right.$$

$$\left.+\int_{\frac{\xi+t}{2}}^{\xi} q(\tau)P_n(\tau,t-\tau+\xi)\,d\tau-\int_{\frac{\xi-t}{2}}^{\xi-t} q(\tau)P_n(\tau,-t-\tau+\xi)\,d\tau\right]d\xi,$$

$$\left|P_n(x,t)\right| \le \left(Q(x)\right)^n \frac{x^{n-1}}{(n-1)!}, \qquad Q(x) = \int_0^x \left|q(\xi)\right| d\xi .$$

Bu baholashlardan (2.17) qator $0 \le t \le x \le \pi$ to'plamda absolyut. va tekis yaqinlashishi kelib chiqadi. Demak,

$$P_1(x,x) = \frac{1}{2}\int_0^x q(t)\,dt, \quad P_{n+1}(x,x) = 0, \quad n \ge 1,$$

$$P(x,t) = \sum_{n=1}^{\infty} P_n(x,t)$$

Bularga asosan (2.13) va (2.16) tengliklarga ega bo‘lamiz.

Almashtirish operatorining umumiy ko'rinishi

E chiziqli normallangan fazo bo’lib, A va B uning E_1 va E_2 qism fazolarida aniqlangan chiziqli operatorlar bo’lsin.

Ta’rif 3.1. O‘zi va teskarisi uzluksiz bo’lgan, $XA = BX$ yoki $A = X^{-1}BX$ shartni qanoatlantiruvchi $X : E_1 \to E_2$ chiziqli operatorga A va B operatorlar uchun almashtirish operatori deyiladi.

Lemma 3.1. *Agar $\varphi \in E_1$ vektor A operatorning λ xos qiymatiga mos keluvchi xos vektori bo’lsa, $\psi = X\varphi$ vektor B operatorning xuddi shu λ xos qiymatiga mos keluvchi xos vektori bo‘ladi.*

Isbot. $B\psi = BX\varphi = XA\varphi = X\lambda\varphi = \lambda X\varphi = \lambda\psi$.

Lemma 3.2. *A. B, C operatorlar mos ravishda E fazoning E_1, E_2, E_3 qism fazolarida aniqlangan operatorlar bo‘lib, A va B operatorlar uchun almashtirish operatori X , B va C operatorlar uchun almashtirish operatori Y bo‘lsa, A va C operatorlar uchun almashtirish operatori Y X bo‘ladi.*

Isbot. Almashtirish operatorining ta’rifiga кo'ra $X A = B X$, $Y B = C Y$ bo’ladi. Ikkinchi tenglikdan $B = Y^{-1}CY$ ifodani topib, birinchisiga qo'ysak; ushbu $XA = Y^{-1}CY\ X$ tenglikka ega bo’lamiz, ya’ni $(Y X)A = C (Y X)$ tenglik kelib chiqadi.

Endi yarim o‘qda berilgan Shturm-Liuvill operatori uchun almashtirish operatorining ko'rinishini topish bilan shug‘ullanamiz.

$E = C^1[0,\infty)$ bo’lib, A va B operatorlar quyidagi

$$A = -\frac{d^2}{dx^2} + q_1(x), \qquad 0 \le x < \infty, \tag{3.1}$$

$$B = -\frac{d^2}{dx^2} + q_2(x), \qquad 0 \le x < \infty, \tag{3.2}$$

ko’rinishga ega bo’lsin. Bu yerda $q_1(x)$, $q_2(x)$ funksiyalar $[0,\infty)$ otraliqda berilgan uzliksiz funksiyalardir.

E_k orqali E fazodagi $f'(0)=h_k f(0) \quad (k=1,2)$ shartni qanoatlantiruvchi, ikki marta uzluksiz differensiallanuvchi funksiyalar to'plamini belgilaylik. Bu yerda h_1 va h_2 chekli haqiqiy sonlar.

Teorema 3.1. *A va B operatorlarining $X: E_1 \to E_2$ almashtirish operatori mavjud bo'lib, u uchun quyidagi tasvir o'rinli:*

$$Xf(x)=f(x)+\int_0^x K(x,t)f(t)dt. \tag{3.3}$$

Bu yerda K (x. t) yadro quyidagi

$$\frac{\partial^2 K}{\partial x^2}-q_2(x)K=\frac{\partial^2 K}{\partial t^2}-q_1(t)K, \tag{3.4}$$

tenglamani va

$$K(x,x)=h_2-h_1+\frac{1}{2}\int_0^x [q_2(s)-q_1(s)]ds \tag{3.5}$$

$$\left.\left(\frac{\partial K}{\partial t}-h_1 K\right)\right|_{t=0}=0 \tag{3.6}$$

shartlarni qanoatlantiradi. Aksincha, $K(x,t)$ funksiya (3.4) tenglamaning (3.5), (3.6) shartlarni qanoatlantiruvchi yechimi bo'lsa, (3.3) tenglik bilan berilgan X operator A va B chiziqli operatorlar uchun almashtirish operatori bo'ladi.

Isbot. I. $Xf(x)\in E_2$ bo'lgani uchun

$$\left.(Xf)'\right|_{x=0}=h_2\left.(Xf)\right|_{x=0}=$$

$$=h_2\left.\left(f(x)+\int_0^x K(x,t)f(t)dt\right)\right|_{x=0}=h_2 f(0),$$

tenglik o'rinli. bo'ladi. Ikkinchi tomondan, (3.3) tenglikdan hosila olib, ushbu

$$(Xf)'=f'(x)+K(x,x)f(x)+\int_0^x \frac{\partial K}{\partial x}f(t)dt \tag{3.7}$$

tenglikda $x=0$ desak, quyidagi

$$(Xf)'_{x=0}=f'(0)+K(0,0)f(0)=h_1 f(0)+K(0,0)f(0)=$$
$$=(h_1+K(0,0))f(0),$$

ifoda hosil bo'ladi. $f\in E_1$ funksiyaning ixtiyoriy ekanligidan ushbu

$$K(0,0)=h_2-h_1 \tag{3.8}$$

tenglik kelib chiqadi.

Endi $B(Xf)=X(Af)$ shartni ishlatamiz. (3.7) tenglikdan hosila olib,

$$(Xf)'' = f''(x) + f(x)\frac{d}{dx}K(x,x) + K(x,x)f'(x) + \\ + \left.\frac{\partial K}{\partial x}\right|_{t=x} f(x) + \int_0^x \frac{\partial^2 K}{\partial x^2} f(t)\,dt$$

Ushbu

$$B(Xf) = -(Xf)'' + q_2(x)(Xf) = -f''(x) - f(x)\frac{d}{dx}K(x,x) - \\ -K(x,x)f'(x) - \left.\frac{\partial K}{\partial x}\right|_{t=x} f(x) + \\ +q_2(x)f(x) - \int_0^x \left(\frac{\partial^2 K}{\partial x^2} - q_2(x)K\right) f(t)\,dt \qquad (3.9)$$

tenglikni hosil qilamiz. Ushbu

$$\int_0^x K(x,t)f''(t)\,dt = \int_0^x K(x,t)\,df'(t) = \\ = K(x,x)f'(x) - K(x,0)f'(0) - \int_0^x \frac{\partial K(x,t)}{\partial t} df(t) = \\ = K(x,x)f'(x) - h_1 K(x,0)f(0) - \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} f(x) + \\ + \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=0} f(0) + \int_0^x \frac{\partial^2 K(x,t)}{\partial t^2} f(t)\,dt,$$

bo'laklab integrallashni ishlatsak, quyidagi

$$X(Af) = \left[-f''(x) + q_1(x)f(x)\right] + \\ + \int_0^x K(x,t)\left[-f''(t) + q_1(t)f(t)\right]dt = \\ = -f''(x) + q_1(x)f(x) + \left.\left(h_1 K(x,t) - \frac{\partial K(x,t)}{\partial t}\right)\right|_{t=0} f(0) - \\ -K(x,x)f'(x) + \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} f(x) - \int_0^x \left(\frac{\partial^2 K}{\partial t^2} - q_1(t)K\right) f(t)\,dt \qquad (3.10)$$

tenglik kelib chiqadi.

(3.9) va (3.10) ifodalarni bir-biriga tenglab, ushbu

$$-f(x)\frac{d}{dx}K(x,x)-\left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x} f(x)+q_2(x)f(x)-$$
$$-\int_0^x\left(\frac{\partial^2 K}{\partial x^2}-q_2(x)K\right)f(t)dt=\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} f(x)+q_1(x)f(x)+$$
$$+\left.\left(h_1K(x,t)-\frac{\partial K(x,t)}{\partial t}\right)\right|_{t=0} f(0)-\int_0^x\left(\frac{\partial^2 K}{\partial t^2}-q_1(t)K\right)f(t)dt$$

(3.11)

tenglikka ega bo’lamiz. (3.11) tenglikda quyidagi

$$\frac{dK(x,x)}{dx}=\left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x}+\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x}$$

formuladan va $f(x)$ funksiyaning ixtiyoriy ekanligidan foydalansak, hamda $K(0,0)=h_2-h_1$ tenglikni e’tiborga olsak, (3.4), (3.5). (3.6) tengliklar kelib chiqadi.

Biz *A* va *B* chiziqli operatorlar uchun almashtirish operatori (3.3) ko'rinishda izlansa, uning yadrosi (3.4), (3.5), (3.6) shartlarni qanoatlantirishini isbotladik

$K(x,t)$ funksiya (3.4), (3.5), (3.6) shartlarni qanoatlantirishidan (3.3) tenglik bilan beriladigan operator *A* va *B* operatorlarning almashtirish operatori bo'lishini ko'rsatish uchun qilingan ishlarni teskari tartibda bajarish yetarli.

II. Endi esa, (3.4)+(3.5)+(3.6) masala yechimining mavjudligini va yagonaligini isbotlaymiz. Buning uchun avvalo almashtirish operatorining ikkinchi xossasidan foydalanib, bu masala o'rniga soddaroq masalani qaraymiz, ya’ni umumiylikni buzmagan holda $q_2(x)\equiv 0,\ h_1=0$ yoki $h_2=0$ deb hisoblash mumkin ekanini ko‘rsatamiz:

$$A=-\frac{d^2}{dx^2}+q(x),\qquad B=-\frac{d^2}{dx^2},\qquad C=-\frac{d^2}{dx^2}+r(x),$$

bo’lib,

$$E_1=\{f(x)\in E\mid\ f'(0)=h_1f(0)\},$$
$$E_2=\{f(x)\in E\mid\ f'(0)=0\},$$
$$E_3=\{f(x)\in E\mid\ f'(0)=h_3f(0)\},$$

bo’lsin. Bundan tashqari *A* va *B* operatorlar uchun almashtirish operatori ushbu

$$Xf(x)=f(x)+\int_0^x K_1(x,t)f(t)dt \tag{3.12}$$

ko’rinishda, *B* va *C* operatorlar uchun almashtirish operatori esa quyidagi

$$Yf(x)=f(x)+\int_0^x K_2(x,t)f(t)dt \tag{3.13}$$

ko'rinishda bo'lsin. U holda *A va*, *C* operatorlar uchun almashtirish operatori ushbu

$$(YX)f(x)=Y\left\{f(x)+\int_0^x K_1(x,t)f(t)dt\right\}=$$
$$f(x)+\int_0^x K_1(x,t)f(t)dt+$$
$$+\int_0^x K_2(x,t)\left[f(t)+\int_0^t K_1(t,s)f(s)ds\right]dt=$$
$$=f(x)+\int_0^x\left[K_1(x,t)+K_2(x,t)+\int_t^x K_2(x,s)K_1(s,t)ds\right]f(t)dt=$$
$$=f(x)+\int_0^x K_3(x,t)f(t)dt$$

ko‘rinishda bo’ladi. Bu yerda

$$K_3(x,t)=K_1(x,t)+K_2(x,t)+\int_t^x K_2(x,s)K_1(s,t)ds.$$

III. Demak. (3.4)+(3.5)+(3.6) masalani tekshirishni $q_2(x)\equiv 0$ va h_1 yoki h_2 nolga teng bo’lgan holda olib borish yetarli. Biz $h_2=0$ bo‘lgan holni ko'rib chiqamiz. $h_1=0$ bo'lgan holda ham xuddi shunday bo’ladi. Ushbu

$$\frac{\partial^2 K}{\partial x^2}=\frac{\partial^2 K}{\partial t^2}-q(t)K \qquad (3.14)$$

$$K|_{t=x}=-h-\frac{1}{2}\int_0^x q(s)ds \qquad (3.15)$$

$$\left.\left(\frac{\partial K(x,t)}{\partial t}-hK(x,t)\right)\right|_{t=0}=0 \qquad (3.16)$$

masalaning yechimi mavjudligini va yagonaligini ko'rsatishimiz kerak.

Agar bu masalada ushbu $\xi=x+t,\quad \eta=x-t$ almashtirishni bajarsak, quyidagi

$$K_x=K_\xi+K_\eta$$
$$K_{xx}=\left(K_\xi+K_\eta\right)_\xi+\left(K_\xi+K_\eta\right)_\eta=K_{\xi\xi}+2K_{\xi\eta}+K_{\eta\eta}$$
$$K_t=K_\xi-K_\eta$$
$$K_{tt}=\left(K_\xi-K_\eta\right)_\xi-\left(K_\xi-K_\eta\right)_\eta=K_{\xi\xi}-2K_{\xi\eta}+K_{\eta\eta}$$

formuladan, ushbu

$$K_{\xi\eta}=-\frac{1}{4}q\left(\frac{\xi-\eta}{2}\right)K,$$

$$K\big|_{\eta=0} = -h - \frac{1}{2}\int_0^{\frac{\xi}{2}} q(s)\,ds$$

$$\left(K_\xi - K_\eta - hK\right)\Big|_{\eta=\xi} = 0$$

tengliklarni hosil qilamiz.

Endi esa oxirgi masalaga ekvivalent bo'lgan integral tenglama tuzamiz. Buning uchun oxirgi masalada quyidagi

$$K(x,t) = K\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = A(\xi,\eta)$$

belgilashni kiritib, uni

$$A_{\alpha\beta} = -\frac{1}{4}q\left(\frac{\alpha-\beta}{2}\right)A \qquad (3.17)$$

$$A(\alpha,0) = -h - \frac{1}{2}\int_0^{\frac{\alpha}{2}} q(s)\,ds \qquad (3.18)$$

$$\left(A_\alpha - A_\beta - hA\right)\Big|_{\beta=\alpha} = 0 \qquad (3.19)$$

ko'rinishda yozib olamiz, hamda (3.17) tenglikni $[0,\eta]$ oraliqda β bo'yicha integrallaymiz:

$$A_\alpha(\alpha,\eta) - A_\alpha(\alpha,0) = -\frac{1}{4}\int_0^{\eta} q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)\,d\beta \qquad (3.20)$$

(3.18) tenglikka asosan $A_\alpha(\alpha,0) = -\frac{1}{4}q\left(\frac{\alpha}{2}\right)$ bo'lgani uchun ushbu

$$A_\alpha(\alpha,\eta) = -\frac{1}{4}q\left(\frac{\alpha}{2}\right) - \frac{1}{4}\int_0^{\eta} q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)\,d\beta \qquad (3.21)$$

tenglik o'rinli bo'ladi. (3.21) tenglikni $[\eta,\xi]$ oraliqda α bo'yicha integrallab, ushbu

$$A(\xi,\eta) - A(\eta,\eta) =$$

$$= -\frac{1}{4}\int_\eta^{\xi} q\left(\frac{\alpha}{2}\right)d\alpha - \frac{1}{4}\int_\eta^{\xi}\left\{\int_0^{\eta} q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)\,d\beta\right\}d\alpha \qquad (3.22)$$

ayniyatga ega bo'lamiz.

Endi $A(\eta,\eta)$ ni hisoblaymiz. (3.19) tenglikka ko'ra ushbu

$$2A_\alpha\big|_{\beta=\alpha} = A_\alpha\big|_{\beta=\alpha} + A_\alpha\big|_{\beta=\alpha} = A_\alpha\big|_{\beta=\alpha} +$$

$$+\left(A_\beta + hA\right)_{\beta=\alpha} = \left(A_\alpha + A_\beta + hA\right)\Big|_{\beta=\alpha} = \frac{dA(\alpha,\alpha)}{d\alpha} +$$

$$+hA(\alpha,\alpha)=e^{-h\alpha}\left(e^{h\alpha}A(\alpha,\alpha)\right)' \qquad (3.23)$$

ayniyat bajariladi. (3.21) va (3.23) tengliklardan quyidagi

$$\left(e^{h\alpha}A(\alpha,\alpha)\right)'=-\frac{1}{2}e^{h\alpha}\left\{q\left(\frac{\alpha}{2}\right)+\int_0^\alpha q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\} \qquad (3.24)$$

formula kelib chiqadi. (3.24) ayniyatni $[0,\eta]$ oraliqda α bo'yicha integrallaymiz va (3.18) shartdan foydalanib, quyidagi

$$A(\eta,\eta)=-he^{-h\eta}-$$
$$-\frac{1}{2}e^{-h\eta}\int_0^\eta e^{h\alpha}\left\{q\left(\frac{\alpha}{2}\right)+\int_0^\alpha q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\}d\alpha, \qquad (3.25$$

tenglikni keltirib chiqaramiz. (3.25) ifodani (3.22) tenglikka qo‘yib, ushbu

$$A(\xi,\eta)=-he^{-h\eta}-\frac{1}{4}\int_\eta^\xi q\left(\frac{\alpha}{2}\right)d\alpha-\frac{1}{2}e^{-h\eta}\int_0^\eta e^{h\alpha}q\left(\frac{\alpha}{2}\right)d\alpha-$$
$$-\frac{1}{2}e^{-h\eta}\int_0^\eta e^{h\alpha}\left\{\int_0^\alpha q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\}d\alpha-$$
$$-\frac{1}{4}\int_\eta^\xi\left\{\int_0^\eta q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\}d\alpha \qquad (3.26)$$

ayniyatga ega bo’lamiz, ya’ni $A(\alpha,\beta)$ funksiya (3.26) integral tenglamani qanoatlantirar ekan. Aksincha, $A(\alpha,\beta)$ funksiya (3.26)' integral tenglamani qanoatlantirsa va $q(x)$ funksiya uzluksiz differensiallanuvchi bo’lsa, (3.26) integral tenglamadan foydalanib, $A(\alpha,\beta)$ funksiya (3.17) + (3.18)+(3.19) masalaning yechimi bo'lishini to‘g‘ridan to‘g‘ri tekshirib ko'rish mumkin.

(3.26) tenglama Volterra turidagi integral tenglama bo‘lgani uchun uning yechimi mavjud va yagonadir. Buni ketma-ket yaqinlashishlar usuli bilan ko‘rsatish mumkin.

$q(x)$ funksiya uzluksiz differensiallanuvchi bo'lmasa, uni uzluksiz differensiallanuvchi $q_n(x)$ funksiyalar bilan yaqinlashtirib, $K_n(x,t)$ funksiyalar ketma-ketligini hosil qilamiz, bu ketmaketlikning limiti $K(x,t)$ almashtirish operatorining yadrosi bo‘ladi

Izoh 3.1. Shuni aytib o'tish kerakki, $K(x,t)$ yadro λ parametrga bog’liq emas.

Misol 1. Agar, teorema 3.1 ning shartida *A* va *B* operatorlar quyidagi

$$A=-\frac{d^2}{dx^2}, \quad h_1=0,$$

va

$$B=-\frac{d^2}{dx^2}+q(x), \quad h_2=h,$$

ko'rinishda berilgan bo'lsa , almashtirish operatori ushbu

$$Xf(x) = f(x) + \int_0^x K(x,t) f(t) dt$$

ko'rinishda bo'ladi. Bu yerda $K(x,t)$ yadro ushbu

$$\begin{cases} K_{xx} - q(x)K = K_{tt}, \\ K(x,x) = h + \frac{1}{2}\int_0^x q(s)ds, \\ K_t\big|_{t=0} = 0, \end{cases} \tag{3.27}$$

masalaning yechimidir.

$\varphi(x,\lambda)$ va $\varphi_0(x,\lambda)$ orqali mos ravishda quyidagi

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 1, \\ y'(0) = h, \end{cases} \quad \text{va} \quad \begin{cases} -y'' = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases}$$

Koshi masalasining yechimlarini belgilaylik. Bizga ma'lumki

$$\varphi_0(x,\lambda) = \cos\sqrt{\lambda}x$$

bo'ladi. Almashtirish operatorining xossasiga ko'ra ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \tag{3.28}$$

tenglik o'rinli bo'ladi.

Misol 2. Agar teorema 3.1 ning shartida *A* va *B* operatorlar ushbu

$$A = -\frac{d^2}{dx^2} + q(x), \quad h_1 = h,$$

va

$$B = -\frac{d^2}{dx^2}, \quad h_2 = 0,$$

ko'rinishda berilgan bo'lsin desak, almashtirish operatori quyidagi

$$Xf(x) = f(x) + \int_0^x H(x,t) f(t) dt$$

ko'rinishda bo'ladi. Bu yerda $H(x,t)$ yadro ushbu

$$\begin{cases} H_{xx} = H_{tt} - q(t)H, \\ H(x,x) = -h - \frac{1}{2}\int_0^x q(s)\,ds, \\ \left(H_t - hH\right)\big|_{t=0} = 0, \end{cases} \tag{3.29}$$

masalaning yechimidir.

Almashtirish operatorining hossasiga ko‘ra ushbu

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)\,dt \tag{3.30}$$

tenglik bajariladi.

Izoh 3.2. Bu yerda ham $H(x,t)$ yadro λ parametrga bog’liq emas.

Keyinchalik

$$q(x) = 2\frac{dK(x,x)}{dx} \tag{3.31}$$

$$q(x) = -2\frac{dH(x,x)}{dx} \tag{3.32}$$

va

$$h = K(0,0) = -H(0,0) \tag{3.33}$$

bog’lanishlar muhim ahamiyat. kasb etadi.

Teskari masalaning qo’yilishi. Yagonalik teoremalari.

Mazkur paragrafda spektral analiz teskari masalasining qo'yilishi va yagonalik teoremalarini isbotlash usullari bilan tanishamiz. Bu usullarning tatbiqiy ahamiyati juda keng bo‘lgani uchun ulardan spektral analizning har xil turdagi teskari masalalarini o’rganishda foydalanish mumkin.

Shturm -Liuvill operatori spektral nazariyasining teskari masalasi ilk bor V .A .Ambarsuinyan [6] tomonidan o’rganilgan.

Ushbu

$$-y''+q(x)y=\lambda y,\quad y'(0)=0,\quad y'(\pi)=0, \tag{4.1}$$

Shturm -Liuvill chegaraviy masalasini qaraymiz. Bu yerda $q(x)\in C[0,\pi]$ haqiqiy uzluksiz funksiya.

Agar (4.1) chegaraviy masalada $q(x)\equiv 0,\ x\in[0,\pi]$ bo'lsa, u holda $\lambda_n=n^2,\ n\geq 0$ bo'lishi ravshan.

Teorema 4.1 *(V .A. Ambarsumyan). Agar (4.1) chegaraviy masalaning xos qiymatlari uchun, ushbu*

$$\lambda_n=n^2,\ n\geq 0 \tag{4.2}$$

tenglik bajarilsa, u holda $q(x)\equiv 0,\ x\in[0,\pi]$ *bo'ladi.*

Isbot. (4.1) chegaraviy masalaning xos qiymatlari uchun

$$\sqrt{\lambda_n}=n+\frac{c_0}{n}+\frac{\gamma_n}{n},\quad \{\gamma_n\}\in l_2, \tag{4.3}$$

asimptotik formulaning o'rinli. Bu yerda

$$c_0=\frac{1}{2\pi}\int_0^\pi q(x)dx. \tag{4.4}$$

(4.2) va (4.3) tengliklarni tenglashtirib,

$$c_0=0,\quad \int_0^\pi q(x)dx=0 \tag{4.5}$$

ekanini topamiz. (4.1) chegaraviy masalaning $\lambda_0=0$ eng kichik xos qivmatiga mos keluvchi xos funksiya $y_0(x)$ bo'lsa u holda

$$y_0''+q(x)y_0=0,\quad y_0'(0)=0,\quad y_0'(\pi)=0, \tag{4.6}$$

bo‘ladi. Ossilyatsiya teoremasiga asosan $y_0(x)$ funksiya $(0,\pi)$ oraliqda nolga ega emas. Agar $y_0(0)=0$ yoki $y_0(\pi)=0$ bo’lsa chegaraviy shartlardan $y_0(x)\equiv 0$ ziddiyat kelib chiqadi. Demak, $y_0(x)\neq 0,\ \ x\in[0,\pi]$.

Ushbu

$$0=\int_0^\pi q(x)dx=\int_0^\pi \frac{y_0''(x)}{y_0(x)}dx=\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx+\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)'dx=$$

$$=\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx+\left.\frac{y_0'(x)}{y_0(x)}\right|_{x=0}^{x=\pi}=$$

$$=\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx+\frac{y_0'(\pi)}{y_0(\pi)}-\frac{y_0'(0)}{y_0(0)}=\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx$$

tenglikdan

$$\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx=0$$

hosil bo’ladi. Oxirgi tenglikdan $y_0'(x)=0$, ya’ni $y_0(x)=const$ kelib chiqadi. (4.6) tenglamadan

$$y_0''(x)-q(x)y_0(x)=0$$

$q(x)=0$ ekanligi kelib chiqadi.

Umuman olganda $\lambda_0,\lambda_1,\lambda_2,...,\lambda_n,...$ xos qiymatlarning, ya’ni spektrning berilishi Shturm - Liuvill chegaraviy masalasini yagona aniqlamaydi. Chunki, ikkita har-xil Shturm -Liuvill chegaraviy masalasi bir xil spektrga ega bo’lishi mumkin.

Masalan, ushbu

$$-y''=\lambda y,\qquad y'(0)=y'(\pi)=0$$

va

$$-y''+\frac{2}{(1+x)^2}y=\lambda y,\quad y'(0)+y(0)=0,\quad y'(\pi)+\frac{1}{\pi+1}y(\pi)=0$$

Shturm -Liuvill chegaraviy masalalari $\lambda_n=n^2,\ n\geq 0$ bir xil spektrga ega.

Quyidagi

$$-y''+q(x)y=\lambda y,\quad x\in[0,\pi] \tag{4.7}$$

$$y'(0)-hy(0)=0 \tag{4.8}$$

$$y'(\pi)+Hy(\pi)=0 \tag{4.9}$$

Shturm -Liuvill chegaraviy masalasi berilgan bo‘lsin. Bu yerda $q(x)\in C[0,\pi]$ haqiqiy uzluksiz funksiya, $h,\ H$ - chekli haqiqiy sonlar.

$\varphi(x,\lambda)$ orqali (4.7) tenglamaning

$$\varphi(0,\lambda)=1,\quad \varphi'(0,\lambda)=h$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

(4.7)-(4.9) Shturm -Liuvill chegaraviy masalasining xos qiymatlar ketma-ketligini $\lambda_n,\ n\geq 0$ orqali belgilaylik. Bu xos qiymatlarga quyidagi $\varphi(x,\lambda_n),\ n\geq 0$ xos funksiyalar mos keladi. Normallovchi o'zgarmaslar ketma-ketligini

$$\alpha_n=\sqrt{\int_0^\pi \varphi^2(x,\lambda_n)dx},\ \ n\geq 0$$

orqali belgilaymiz.

Ta’rif 4.1. Ushbu $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ ketma-ketliklar juftligiga Shturm -Liuvill chegaraviy masalasining spektral xarakteristikalari yoki spektral berilganlari deyiladi.

Ta’rif 4.2. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalar yordamida $q(x)$ funksiyani va $h,\ H$ sonlarni topish masalasiga teskari spektral masala deyiladi.

Yuqoridagi masalani atroflicha o‘rganish maqsadida ikkinchi

$$-y''+\tilde q(x)y=\lambda y,\quad x\in[0,\pi], \tag{4.10}$$

$$y'(0)-\tilde h y(0)=0, \tag{4.11}$$

$$y'(\pi)+\tilde H y(\pi)=0, \tag{4.12}$$

Shturm -Liuvill chegaraviy masalasini qaraymiz. Bu yerda $\tilde q(x)\in C[0,\pi]$ haqiqiy uzluksiz funksiya, $\tilde h, \tilde H$ -chekli haqiqiy sonlar.

$\tilde\varphi(x,\lambda)$ orqali (4.10) tenglamaning

$$\tilde\varphi(0,\lambda)=1,\quad \tilde\varphi'(0,\lambda)=\tilde h,$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

(4.10)-(4.12) chegaraviy masalasining xos qiymatlar ketma-ketligini $\tilde\lambda_n,\ n\geq 0$ orqali, normallovchi o'zgarmaslar ketma-ketligini

$$\tilde\alpha_n=\sqrt{\int_0^{\pi}\tilde\varphi^2\left(x,\tilde\lambda_n\right)dx},\quad n\geq 0,$$

orqali belgilaymiz.

Teorema 4.2 *(V.A.Marchenko). Agar* $\lambda_n=\tilde\lambda_n,\ \alpha_n=\tilde\alpha_n,\ n\geq 0$ *bo‘lsa, u holda ushbu*

$$h=\tilde h,\ H=\tilde H,\ q(x)=\tilde q(x),\ x\in[0,\pi],$$

tengliklar o‘rinli bo‘ladi.

Isbot. Almashtirish operatori haqidagi teorema 3.1 ga asosan

$$\tilde\varphi(x,\lambda)=\varphi(x,\lambda)+\int_0^x K(x,t)\varphi(t,\lambda)\,dt, \tag{4.13}$$

tenglik o'rinli bo'ladi.

Ixtiyoriy $f(x)\in L^2(0,\pi)$ funksiyaning $(f,\tilde{\varphi})$ Furye koeffitsiyentini (4.13) tenglikdan foydalanib hisoblaymiz:

$$\int_0^\pi f(x)\tilde{\varphi}(x,\lambda)\,dx=\int_0^\pi f(x)\left[\varphi(x,\lambda)+\int_0^x K(x,t)\varphi(t,\lambda)\,dt\right]dx=$$

$$=\int_0^\pi f(x)\varphi(x,\lambda)\,dx+\int_0^\pi f(x)\left[\int_0^x K(x,t)\varphi(t,\lambda)\,dt\right]dx=$$

$$=\int_0^\pi f(x)\varphi(x,\lambda)\,dx+\int_0^\pi\left[\int_x^\pi K(t,x)f(t)\,dt\right]\varphi(x,\lambda)\,dx=$$

$$=\int_0^\pi\left[f(x)+\int_x^\pi K(t,x)f(t)\,dt\right]\varphi(x,\lambda)\,dx=\int_0^\pi g(x)\varphi(x,\lambda)\,dx. \qquad (4.14)$$

Bu yerda

$$g(x)=f(x)+\int_x^\pi K(t,x)f(t)\,dt \qquad (4.15)$$

(4.14) tenglikda $\lambda=\tilde{\lambda}_n,\ n=0,\ 1,\ 2,\ \dots$ deb olsak, u holda ushbu

$$\tilde{a}_n=\frac{1}{\tilde{\alpha}_n}\int_0^\pi f(x)\tilde{\varphi}(x,\tilde{\lambda}_n)\,dx=\frac{1}{\alpha_n}\int_0^\pi f(x)\tilde{\varphi}(x,\lambda_n)\,dx,$$

$$a_n=\frac{1}{\alpha_n}\int_0^\pi g(x)\varphi(x,\lambda_n)\,dx$$

Furye koeffitsiyentlari uchun $\tilde{a}_n=a_n,\ n=0,\ 1,\ 2,\ \dots$ tenglik bajariladi. Parseval tengligiga asosan

$$\int_0^\pi |f(x)|^2\,dx=\sum_{n=0}^\infty |\tilde{a}_n|^2=\sum_{n=0}^\infty |a_n|^2=\int_0^\pi |g(x)|^2\,dx,$$

kelib chiqadi, ya'ni

$$\|f\|_{L^2}=\|g\|_{L^2}. \qquad (4.16)$$

Ushbu

$$Af(x) = f(x) + \int_x^{\pi} K(t,x) f(t) dt$$

operatorni qaraylik. U holda (4.15) ga asosan

$$Af(x) = g(x),$$

bo'ladi. (4.16) tenglikdan esa

$$\|Af\|_{L^2} = \|f\|_{L^2}$$

kelib chiqadi. Bu esa A operatorning $L^2(0,\pi)$ fazoda unitarligini bildiradi. Unitar operatorlar uchun $A^*A = I$ tenglik o'rinli bo'ladi.

A^* operatorni topish qiyin emas:

$$A^*h(x) = h(x) + \int_0^x K(x,t) h(t) dt$$

$A^*\{Af(x)\}$ ning aniq ifodasini topamiz:

$$A^*\{Af(x)\} = Af(x) + \int_0^x K(x,t)\{Af(t)\} dt =$$

$$= f(x) + \int_x^{\pi} K(t,x) f(t) dt +$$

$$+\int_0^x K(x,t)\left\{ f(t) + \int_t^{\pi} K(s,t) f(s) ds \right\} dt =$$

$$= f(x) + \int_x^{\pi} K(t,x) f(t) dt + \int_0^x K(x,t) f(t) dt +$$

$$+\int_0^x \left\{ \int_t^\pi K(x,t)K(s,t)f(s)ds \right\} dt$$

Bu yerda integrallash tartibini almashtirib quyidagi

$$A^*\{Af(x)\} = f(x) + \int_0^x \left\{ K(x,t) + \int_0^t K(x,s)K(t,s)ds \right\} f(t)dt +$$

$$+\int_x^\pi \left\{ K(t,x) + \int_0^x K(x,s)K(t,s)ds \right\} f(t)dt$$

formulaga ega bo'lamiz.

$A^*\{Af(x)\} = f(x)$ ayniyatdan foydalansak

$$\int_0^x \left\{ K(x,t) + \int_0^t K(x,s)K(t,s)ds \right\} f(t)dt +$$

$$+\int_x^\pi \left\{ K(t,x) + \int_0^x K(x,s)K(t,s)ds \right\} f(t)dt = 0$$

tenglik kelib chiqadi. Bu yerda

$$f(t) = \begin{cases} K(x,t) + \int_0^t K(x,s)K(t,s)ds, & t \in [0,x], \\ 0, & t \in (x,\pi), \end{cases}$$

deb olsak, ushbu

$$\int_0^x \left\{ K(x,t) + \int_0^t K(x,s)K(t,s)ds \right\}^2 dt = 0$$

tenglik kelib chiqadi. Bunga ko'ra

$$K(x,t) + \int_0^t K(x,s)K(t,s)ds = 0$$

Oxirgi tenglik x ning har bir tayinlangan qiymatida $K(x,t)$ funksiyaga nisbatan bir jinsli Volterra integral tenglamasidir. Bunday tenglama faqat nol yechimga ega bo‘lishidan $K(x,t) \equiv 0 \ (t \le x)$ kelib chiqadi. Buni (4.13) formulaga qo'ysak

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda)$$

ayniyat xosil bo’ladi. Boshlang’ich va chegaraviy shartlarga ko‘ra

$$h = \tilde{h} \text{ va } H = \tilde{H}$$

Ushbu

$$-\varphi'' + q(x)\varphi = \lambda\varphi$$

$$-\varphi'' + \tilde{q}(x)\varphi = \lambda\varphi$$

tengliklardan foydalanib,

$$\left[q(x) - \tilde{q}(x)\right]\varphi(x,\lambda) = 0$$

bo'lishini topamiz. Bundan va $q(x)$, $\tilde{q}(x)$ funksiyalarning uzluksizligini hamda $\varphi(x,\lambda)$ funksiyaning nollarini ajralganligini e’tiborga olsak, $q(x) = \tilde{q}(x)$ ayniyat kelib chiqadi.

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