IMPROVED LACTATION CURVE MODEL

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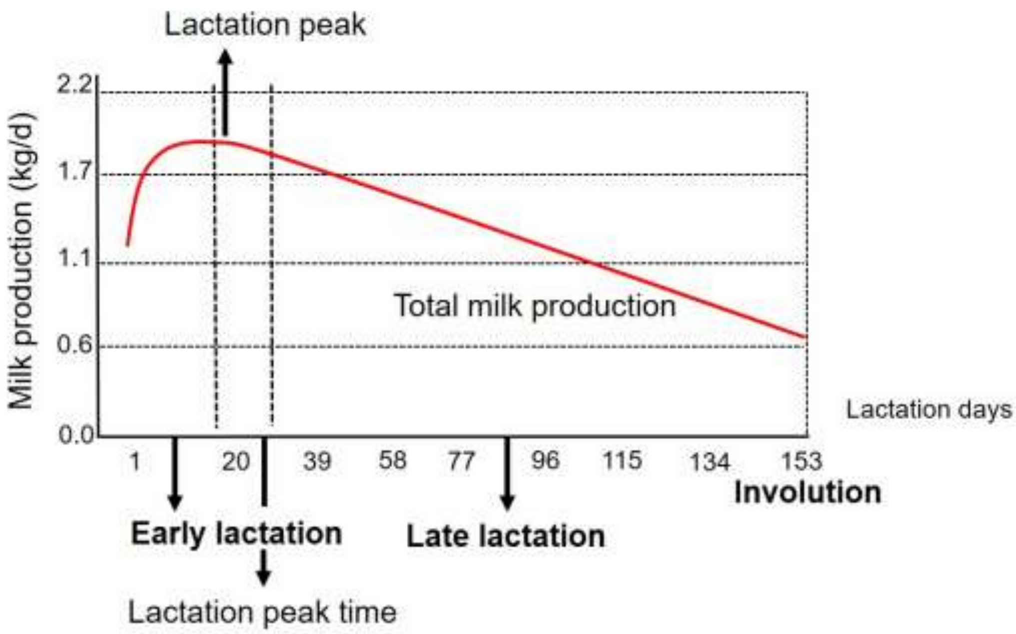
ABSTRACT:

This article analyzes mathematical models of the lactation curve (LC), which describe variations in the milk yield of dairy cows. Special attention is given to the widely used Wood and Wilmink models, with an in-depth discussion of their structure, biological interpretation, and parameter effects. Considering the advantages and limitations of both models, a new improved model is proposed that combines the main curve of the Wood model with the short-term exponential component of the Wilmink model. The proposed model adapts better to initial conditions and provides a highly accurate representation of the LC. Its practical effectiveness is evaluated using 305-day real milk yield data from 300 cows, and compared with the Wood and Wilmink models using RMSE and R^2 metrics. The results show that the new model achieves higher accuracy in predicting milk yield.

INTRODUCTION. Lactation refers to the period from an animal's parturition (giving birth) until it is dried off. The duration of lactation is approximately 305 days. During this period, the daily milk yield undergoes significant fluctuations. A lactation curve (LC) is a graphical representation of changes in milk productivity over the lactation period (see Figure 1). The LC provides invaluable information for evaluating the genetic and management performance of dairy cows. Typically, the LC rises rapidly until it reaches a peak and then gradually declines until the dry period begins. Characterized by peak yield, time to peak, and persistence, the LC is a key indicator used not only for assessing milking potential and making breeding decisions, but also for optimizing feeding and health management. For large groups of cows, the average LC follows a typical shape, which can be described using mathematical models ranging from simple to complex equations.

Various equations have been proposed to construct the LC, most of which fit well to the average LC observed in large cow populations [1].



1-rasm. Figure 1. Example of a lactation curve

To date, more than 40 models have been proposed to describe the lactation curve (see Table 1). These models can be conditionally divided into three groups based on their development approach: empirical, mechanistic (logical), and semi-empirical models. Among them, the models developed by Wood, Wilmink, Dijkstra, Ali and Schaeffer, Bordy, and Mikailsoy Fariz are considered the most widely used.

The **Wood model** is one of the most well-known empirical models used to describe the time-dependent variation in daily milk yield of cows [1]. This model has the following functional form:

$$Y(t) = a \cdot t^b \cdot e^{-c \cdot t}$$

Here $Y(t)$ is the milk yield on day t of lactation, and a, b, and c are model parameters. The Wood equation essentially takes the form of an incomplete gamma function, where the multiplier t^b represents the initial growth rate at the beginning of lactation, and $e^{-c \cdot t}$ reflects the subsequent decline rate [1, 2].

№	year	author	model
	1923	Brody	$y = ae^{-ct}$
	1924	Brody	$y = ae^{-bt} - ae^{-ct}$
	1950	Sikka	$y = ae^{-bt-ct^2}$
	1961	Geyns	$y = ate^{-ct}$
	1966	Nelder	$y = \frac{t}{a + bt + ct^2}$
	1967	Wood	$y = at^b e^{-ct}$
	1971	McNally	$y = at^b e^{-ct+d\sqrt{t}}$
	1971	Dave	$y = a + bt - ct^2$
	1974	Hadeler	$y = a_0 + \sum_{i=1}^{r} a_i t^i, \quad r = 2,3,4$
	1977	Schaffer	$y = \frac{a}{c} e^{-bt}(1 - e^{-ct})$
	1978	Cobby va Le Du	$y = a - bt - ae^{-ct}$
	1979	Molina	$y = \begin{cases} a + bt & t < t_0 \\ a + b(2t_0 - t) & t \geq t_0 \end{cases}$
	1981	Dhanoa	$y = at^{bc} e^{-ct}$
	1982	Singh	$y = a - bt + c \ln t$
	1982	Singh	$y = a + bt + ct^2 + d \ln t$
	1984	Jenkins and Ferrell	$y = ate^{-ct}$
	1984	Bianchini	$y = a + bt + \frac{c}{t}$
	1984	Bianchini	$y = a + bt + ct^2 + d \ln t$
	1986	Hayashi	$y = a(e^{-\frac{t}{b}} - e^{-\frac{t}{bc}})$
	1986	Grossman	$y = at^b e^{-ct}(1 + u \sin t + v \cos t)$

	1987	Wilmink	$y = a + be^{-kt} + ct$
	1987	Wilmink	$y = a + be^{-kt} + ct + dt^2$
	1987	Ali va Schaffer	$y = a + b\frac{t}{305} + c\left(\frac{t}{305}\right)^2 + d\ln\left(\frac{305}{t}\right) + e\ln^2\left(\frac{305}{t}\right)$
	1989	Cappio-Borlino	$y = a + bt + ct^2 + dt^3$
	1989	Morand and Gnanasakthy	$y = e^{a-bt-ct^2-d/t}$
	1989	Morand and Gnanasakthy	$y = at^b e^{-ct+dt^2}$
	1993	Rook	$y = a\left(\frac{c+t}{b+c+t}\right)e^{-dt}$
	1995	Guo and Swalve	$y = a + b\sqrt{t} + c\ln t$
	1995	Cappio-Borlino	$y = at^{b\exp(-ct)}$
	1997	Dijkstra	$y = a\exp\left[\frac{b}{c}(1 - e^{-ct}) - dt\right]$
	1997	Dijkstra	$y = \mathrm{M}_0\exp\left\{\frac{\mu_\mathrm{T}[1 - \exp(-kt)]}{k - \lambda t}\right\}$
	1997	Guo	$y = a + bt + ct^2 + dt^3 + k\ln t$
	2000	Pollot	$y = a\begin{Bmatrix} 1/[1 + ((1-b)/b)e^{-ct}] - \\ -1/[1 + ((1-d)/d)e^{-gt}] \end{Bmatrix}(1 - e^{-ht})$
	2005	Dag	$y = a + bt + ct^2 + dt^3$
	2010	Keskin	$y = at^b e^{-ct-\frac{d}{t}}$
	2011	Kitikov	$y = \begin{cases} a_1 + b_1 t + c_1 t^2 & ,1 \le t < t_{max} \\ at^b e^{-ct} & ,t_{max} \le t \le t_k \end{cases}$ t_{max}-maksimal sut mahsuldorlik kuni t_k laktatsiya davrining oxirgi kuni
	2012	Kutsenko	$y = \frac{A}{1 + e^{at+b}} + \frac{q}{2}t^2 + c$
	2013	Mikaylov	$y = at^b e^{-ct-dt^2-q/t}$

	2013	Mikaylov	$y = a + bt + ct^2 + dt^3 + k\ln t + q(\ln t)^2 + r(\ln t)^3$
	2020	Mikoilsoy	$y = at^b e^{-ct-dt^2-\frac{q}{t}} \cdot \cos(kt)$
	2020	Mikoilsoy	$y(t) = \frac{K}{1+\exp(a+bt)} + c + dt + qt^s$

This model is based on the assumption that a cow's milk yield during lactation initially increases, reaches a certain peak, and then gradually declines. Therefore, the lactation curve in the Wood model consists of three phases: a growth phase, a peak, and a decline phase [2].

Growth phase: The presence of the multiplicative term t^b represents the initial increase in milk production. If $b > 0$, the value of $Y(t)$ increases rapidly in proportion to t^b when t is small. Physiologically, this corresponds to the increase in the number and activity of mammary gland cells at the beginning of lactation. The parameter b determines the curvature of the graph at the start of the curve. For example, when $0 < b \leq 1$, the curve takes on the typical shape of rising initially and then declining. If $b = 0$, the model reduces to a simple exponential decline, and $b < 0$, it represents only a decreasing trend [2].

Peak point: In the Wood model, the maximum milk yield occurs at time $t = \frac{b}{c}$ [1,2]. This expression is obtained by setting the derivative of the model equal to zero, i.e., $\frac{dY}{dt} = 0$. The milk yield at this peak time is determined using the following equation [1]:$Y_{\text{peak}} = a\left(\frac{b}{c}\right)^b e^{-c\cdot\left(\frac{b}{c}\right)} = a\left(\frac{b}{c}\right)^b e^{-b}$.

Decline phase: The multiplier e^{-ct} represents the exponential decline in milk production over time. The parameter c determines the rate of this decline [1]. For example, c can be compared to the decay constant in radioactive disintegration - the larger the value of c, the faster the milk yield decreases.

In his research, Wood demonstrated that the gamma probability density function can be reparameterized to describe the lactation curve [4]. In this formulation, the shape parameter b of the gamma function corresponds to the initial increase in the milk yield curve, while $1/c$ reflects the rate of decline. Therefore, the Wood model has been noted for its ability to accurately describe the lactation curves of many cow breeds and for its flexibility [2].

The Wilmink model is also a widely used, simple nonlinear model for describing the lactation curve [3]. It has the following functional form [5]:

$$Y(t) = a + b \cdot e^{-kt} + c \cdot t$$

Here a, b and c are empirical parameters, and k is the exponential decay rate. Unlike the Wood model, the algebraic structure of the Wilmink model consists of additive components.

The role of each term in the Wilmink equation is interpreted as follows: a represents the asymptotic or baseline level of the model. As time progresses and the exponential term decays, $Y(t)$ approaches approximately $a + c$. Therefore, the parameter a reflects, to some extent, the maximum milk production capacity [3]. If c is close to zero, then a provides a stable high level of milk yield.

The term $b \cdot e^{-kt}$ models the rapid change at the beginning of lactation. At $t = 0$, the value of this term equals b, and over time it decreases exponentially in the form of $b \cdot e^{-kt}$. This component represents the early phase of the lactation curve. For example, if b is negative, the initial effect of the $b \cdot e^{-kt}$ term is also negative, causing $Y(t)$ to start at a lower level. As this exponential effect diminishes over time, $Y(t)$ increases. From a biological perspective, the $b \cdot e^{-kt}$ term can reflect the negative influence of factors such as postpartum stress, hormonal changes, or negative energy balance on milk production - effects that gradually weaken over time. Conversely, if b is positive, the model may represent a lactation curve that starts with high milk yield and then declines. Therefore, scientific literature often notes that for the Wilmink model to produce a typical "rise–peak–decline" lactation curve, it is necessary for both $b < 0$ and $c < 0$ [6].

The term $c \cdot t$ (linear) represents the long-term trend that occurs during lactation. If c is positive, the model shows an increasing trend over time. Typically, c is negative, so the $c \cdot t$ term models a linear decline in milk yield after the peak. In fact, the exponential term becomes nearly zero around day 50– , after which $Y(t)$ mainly follows a linear trend of the form $a + c \cdot t$Thus, a negative c value leads to a decline in milk production toward the end of lactation. In the Wilmink model, the more negative the parameter c , the faster the milk yield decreases.

From the perspective of lactation phases, the Wilmink model is interpreted as follows:

Initial **growth** phase – Due to the be^{-kt} term, $Y(t)$ starts at a lower value at the beginning of lactation (if $b < 0$) and increases rapidly. As the exponential term fades, milk production approaches $a + c \cdot t$, marking the end of the initial rise. Thus, the exponential component governs changes in milk yield during the first 1–2 months of lactation.

Peak and **mid-lactation** phase – Although the exact peak point in the model can be determined analytically, intuitively, the peak occurs when the declining exponential term and the increasing linear term balance each other. Typically, when b and c are both chosen to be negative, the model exhibits a single maximum point. The peak usually occurs between the 4th and 8th weeks of lactation, depending on the values of these parameters. The parameter a defines the general level around the peak - some researchers interpret a as the “milk yield level near the peak.”

Decline phase – After 2–3 months, the exponential component becomes nearly zero, and $Y(t)$ transitions to the form $a + ct$. If $c < 0$, this indicates an approximately linear decrease in milk yield. In reality, the lactation curve may show a slightly accelerated decline in the final months, but the $c \cdot t$ term in the Wilmink model approximates this decline in a smooth, nearly linear fashion. As a result, the parameter c becomes a key factor in controlling lactation persistency - how long milk production remains after the peak. A small negative value of c indicates a longer and more stable milk production period, while a larger negative c implies a more rapid decline.

The popularity of the Wilmink model is attributed to its simplicity and practicality it contains only 3 or 4 parameters and can be easily fitted using standard regression methods. Numerous studies in the scientific literature have shown that this model fits daily milk yield data of cows quite well [3].

Improved model. Each of the models discussed above Wood and Wilmink - has its own advantages and limitations. The Wood model, as an incomplete gamma function, effectively captures the typical shape of the lactation curve. However, because it is bound to a single formula, its flexibility may be limited in certain cases (for example, when representing values that are unexpectedly low or high at the beginning of lactation, or when independently controlling the rate of decline after the peak). The Wilmink model, on the other hand, provides more flexibility in shaping the curve through its additive components. However, it may not represent long-term physiological processes as accurately due to its linear nature. Moreover, in the Wilmink model, the parameter a is treated as a baseline value and is added independently to the function. This can sometimes result in less accurate fitting to real data compared to the Wood model.

For this reason, in order to more accurately represent the lactation curve and combine the strengths of both models, an improved model is proposed.

This improved model is constructed as a combination of the Wood and Wilmink functions and takes the following form:

$$Y(t) = \alpha \cdot a \cdot t^{b} \cdot e^{-ct} + d \cdot e^{-kt}$$

Here, the first term $\alpha \cdot a \cdot t^b \cdot e^{-ct}$ follows the functional form of the Wood model, while the second term $d \cdot e^{-kt}$ resembles the exponential component of the Wilmink model.

The main idea behind the construction of the improved model is to consider the lactation process as a sum of several sub-processes. From a biological standpoint, a cow's milk production may consist of several independent components:

Primary lactation process – This is represented by the Wood component, which models the proliferation of mammary gland cells after calving and their subsequent decline in activity. The term $\alpha \cdot a \cdot t^b \cdot e^{-ct}$ represents this process. Its parameters (a, b, c) define the timing of the peak, the peak level, and the rate of decline, respectively, as in the original Wood model.

Auxiliary process – This additional component is represented by $d \cdot e^{-kt}$. It is a rapidly decaying exponential term that plays an important role during the early days of lactation. In the initial phase, a cow's milk yield may pass through an anomalous stage due to factors such as the postpartum colostrum period, hormonal changes in the body, stress, etc. These short-term effects may not align perfectly with the main lactation curve. If such effects are not accounted for, the Wood model alone may fail to fit the early data accurately. In the improved model, the $d \cdot e^{-kt}$ term serves to capture these short-term influences. If necessary, d can be assigned a negative value, making this term represent a temporary reduction in milk yield. Whether d is positive or negative is determined empirically during the model fitting process.

In the scientific literature, researchers such as Grossman and Koops have modeled the lactation curve as a sum of multiple gamma functions, separately representing the first and second phases of lactation [7]. Similar approaches have been proposed in other studies as well. In her research, Sitkowska identified the Wood and Wilmink models as among the best-performing models [9]. Some studies have also highlighted the potential of combining these models to construct a more flexible lactation curve [10]. The use of the improved model allows for a more accurate representation of the appropriate curve shape for lactation and enables the description of various curve types. This is important because certain cows exhibit atypical lactation patterns, for example, a continuous decline in milk yield or the presence of two distinct peaks. The additional exponential term in the model provides extra flexibility and can capture deviations from the standard gamma-shaped curve. The proposed model, which combines the Wood and Wilmink functions, offers a similar capability but in a simplified form.

The purpose of introducing the α coefficient into the improved model is to mathematically integrate the Wood and Wilmink formulas consistently and to adapt the model to real initial conditions. As noted earlier, in the Wood model, the predicted milk yield on day $t = 1$ may differ from actual data. For example, there have been cases where the curve fitted using only the Wood model overestimated the milk yield on the first day of measurement [8]. To address this issue, the α coefficient is defined as follows:

$$\alpha = \frac{Y_{actual}(1) - de^{-k}}{ae^{-c}}$$

As a result, $Y(1) = Y_{\text{actual}}(1)$, meaning that the model curve aligns exactly with the data curve at the first point.

The mathematical rationale behind this approach is the distribution of weight between the Wood and Wilmink components of the model. The coefficient α essentially indicates the weight assigned to the Wood part. If $\alpha < 1$ it means the values computed by the Wood formula are slightly reduced; if $\alpha > 1$, the Wood component is amplified. Meanwhile, the $d \cdot e^{-kt}$ term is weighted independently through the parameter d. As a result, at $t = 1$, the sum of the two components matches the actual observed value.

Introducing the α coefficient helps to more accurately model the beginning of the lactation curve. From a mathematical standpoint, it may appear as if one additional free parameter has been added to the model; however, in practice, α can be directly determined from the initial condition. Therefore, α is typically not treated as an independent parameter. This simplifies the model identification process no separate fitting procedure is required for α as it is sufficient to equate it to the observed value at $t = 1$.

Mathematically, the new improved model contains five parameters (α, b, c, d, k). These parameters allow for the adjustment of the shape of the lactation curve. For example, if a cow's milk yield is very low during the first five days after calving and then increases sharply, the Wood model may not fully capture this behavior. However, in the improved model, this can be accounted for using the parameters d and k.

To evaluate the practical effectiveness of the newly developed model, comparative experiments were conducted using real milk yield data. The experiment was carried out on a dairy farm with 300 lactating cows. For each cow, daily milk yield (kg/day) was recorded over a 305-day lactation period. The goodness-of-fit for each model was analyzed using the following statistical metrics (see Table 1): RMSE (Root Mean Squared Error) and R^2 (Determination Coefficient)

Table 1. Comparative analysis of the Wood, Wilmink, and improved models

	RMSE			**R²**		
Model	**Wood**	**Wilmink**	**Improved model**	**Wood**	**Wilmink**	**Improved model**
1st Lactation	2.836	2.921	**2.728**	0.621	0.592	**0.671**
2nd Lactation	2.812	2.906	**2.694**	0.713	0.636	**0.774**
3rd Lactation	2.747	2.823	**2.637**	0.756	0.712	**0.792**
4th Lactation	2.698	2.733	**2.578**	0.798	0.754	**0.821**
5th Lactation	2.612	2.692	**2.528**	0.816	0.796	**0.883**
6th Lactation	2.566	2.649	**2.458**	0.824	0.811	**0.902**
7th Lactation	2.447	2.525	**2.321**	0.865	0.827	**0.925**

Conclusion. This study examined the mathematical modeling of lactation curves in dairy cows, focusing on two of the most widely used models — the Wood and Wilmink models. While both models are effective in capturing the general dynamics of milk yield over a lactation period, each has specific limitations in accurately representing early lactation anomalies or long-term trends. To address these challenges, an improved hybrid model was proposed by combining the core structure of the Wood model with the short-term exponential component of the Wilmink model.

The improved model introduces an additional coefficient (α) to ensure alignment with the initial milk yield conditions, enhancing the model's accuracy and flexibility without significantly increasing complexity. Experimental validation using real 305-day milk yield data from 300 cows demonstrated that the new model outperforms both the Wood and Wilmink models in terms of RMSE and R^2 metrics across all lactation orders. Overall, the proposed model offers a more precise and adaptable tool for modeling lactation curves, making it useful for both scientific analysis and practical decision-making in dairy herd management.

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