FUNKSIYA LIMITINI TOPISHGA DOIR MISOLLAR

Jumaboyeva Zevarjon Rustamboyevna [1]
[1] *Urganch davlat pedagogika instituti,*
Fizika-matematika kafedrasi o'qituvchisi
jumaboyevazevarjon9@gmail.com

MAQOLA MALUMOTI

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KALIT SO'ZLAR:
funksiya, limit, aniqlanish sohasi, trigonometrik funksiya, eksponensial funksiya, ajoyib limitlar, matematik tahlil.

ANNOTATSIYA:
Mazkur maqolada matematik analizda muhim ahamiyatga ega bo'lgan bir nechta funksiyalarning aniqlanish sohasi, formulalari va ularning xossalari tahlil qilinadi. Har bir funksiya uchun uning aniqlanish sohasi aniq belgilab olingan, ba'zilari esa limitlar nuqtai nazaridan chuqur o'rganilgan. Maqola matematik analiz kursi doirasida talabalar va o'qituvchilar uchun foydali metodik qo'llanma bo'lib xizmat qiladi.

KIRISH. Funksiyalar matematik analizning asosiy tushunchalaridan biridir. Ular real hayotdagi fizik, iqtisodiy va boshqa jarayonlarni modellashtirishda keng qo'llaniladi. Ayniqsa, ularning limitlari va xossalari ko'plab amaliy masalalarni yechishda muhim ahamiyat kasb etadi. Quyida keltirilgan funksiyalar orasida matematik analizda tez-tez duch kelinadigan ajoyib va muhim funksiyalar mavjud.

Bizga biror $\boldsymbol{X}$ $(\boldsymbol{X} \subset \mathbb{R})$ to'plam berilgan bo'lib, $\boldsymbol{a}$ uning limiy nuqtasi bo'lsin. $\boldsymbol{f}(\boldsymbol{x})$ va $\mathbf{g}(\boldsymbol{x})$ funksiyalar $\boldsymbol{a}$ nuqtada chekli limitga ega bo'lib,

$$\lim_{x\to a} f(x) = b, \quad \lim_{x\to a} g(x) = c$$

bo'lsin. U xolda

1) $\lim_{x\to a}[f(x) \pm g(x)] = b \pm c$

2) $\lim_{x\to a}[f(x) \cdot g(x)] = b \cdot c$

3) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{b}{c},\quad c\neq 0$

bo'ladi.

Agar $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=1$ bo'lsa $f(x)$ va $g(x)$ funksiyalar $x\to x_0$ bo'lganda ekvivalent deyiladi va $f(x)\sim g(x)$ kabi belgilnadi. Funksiyaning o'rniga uning ekvivelentidan foydalanish limit hisoblashni ancha osonlashtiradi. $x\to 0$ bo'lganda limitda quyidagi almashtirishlarni qilish mumkin:

$$\sin x\sim x,\quad a^x\sim 1+x\ln a,\quad (1+x)^a\sim 1+ax,\quad \cos x\sim 1-\frac{x^2}{2},\quad 1+x\sim e^x.$$

Ajoyib va muhim limitlar.

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{tgx}{x}=\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}=\lim\limits_{x\to 0}\dfrac{arctgx}{x}\lim\limits_{x\to 0}\dfrac{shx}{x}=\lim\limits_{x\to 0}\dfrac{thx}{x}=1$

2. $\lim\limits_{n\to\infty}\dfrac{\sin\alpha x}{x}=\alpha,\alpha\in\mathbb{R}$

3. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=\lim\limits_{x\to\infty}(1+\dfrac{1}{x})^x=e.$ **4.** $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$

5. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a\quad (a>0).$ **6.** $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\ln e=1.$

7. $\lim\limits_{x\to 0}\dfrac{(1+x)^\alpha-1}{x}=\alpha\quad (\alpha\in\mathbb{R}).$

8. $\lim\limits_{x\to+\infty}x^0\ln x=\lim\limits_{x\to+\infty}x^{-d}\ln x=\lim\limits_{x\to+\infty}x^c e^{-x}=0\quad (a>0).$

Ba'zi bir ajoyib va uhum limitlarning isbotlari:

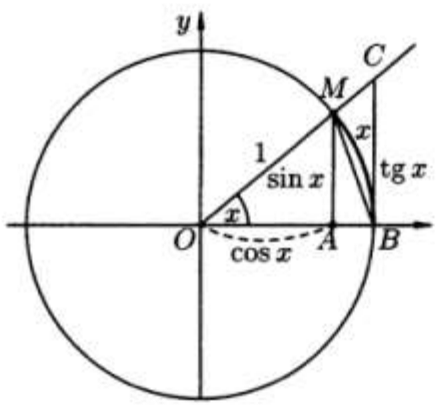
1-ajoyib limit: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

1.1 $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

$\lhd\quad \sin x<x<tgx,\quad \left(0<x<\dfrac{\pi}{2}\right)$ tengsizlik o'rinli, $\boldsymbol{sinx>0}$ bo'lgani uchun

$1<\dfrac{x}{\sin x}<\dfrac{1}{\cos x}$ ko'rinishda yozish mumkin,bundan $0<1-\dfrac{\sin x}{x}<1-\cos x$ tengsizlar kelib chiqadi. Bunda $1-\cos x=2\sin^2\dfrac{x}{2}\le 2\cdot\dfrac{|x|}{2}=|x|$ ko'rinishini

o'zgartirmiz.



$$0<1-\frac{\sin x}{x}<1-\cos x$$

Bu tengsizlikni ko'rinishini o'zgartiramiz:

$$0<\left|1-\frac{\sin x}{x}\right|<|x|$$

Agar $\forall\boldsymbol{\varepsilon}>\mathbf{0}$ son uchun shunday $\boldsymbol{\delta}>\mathbf{0}$ deb ε va $\dfrac{\pi}{2}$ sonlarining kichigi olinsa, argument x ning topilsaki, argument x ning $\mathbf{0}<|\boldsymbol{x}|<\boldsymbol{\delta}$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$\left|1-\frac{\sin x}{x}\right|=\left|\frac{\sin x}{x}-1\right|<\varepsilon$$

tengsizlik kelib chiqadi.Bu esa funksiya limitini Koshi ta'rifiga ko'ra limitning to'g'riligini anglatadi. ⊳

⊲ **1.2** $\lim\limits_{x\to 0}\dfrac{tgx}{x}=1$

$$\lim_{x\to 0}\frac{tgx}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{\cos 0}=1.\rhd$$

$\lhd$ **1.3** $\lim\limits_{x\to 0}\frac{\arcsin x}{x}=1$ $\arcsin x=t$ belgilash kiritib hisoblaymiz: bunda $x=\sin t, x\to 0, \Rightarrow t\to 0$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{t\to 0}\frac{t}{\sin t}=\lim_{t\to 0}\frac{1}{\sin t}\cdot\frac{t}{t}=\frac{1}{\lim\limits_{t\to 0}\frac{\sin t}{t}}=\frac{1}{1}=1.\rhd$$

$\lhd$ **1.4** $\lim\limits_{x\to 0}\frac{\mathrm{arc}\,tgx}{x}=1$ $arctgx=t$ **belgilash kiritib hisoblaymiz: bunda**

$x=tgt, x\to 0, \Rightarrow t\to 0$

$$\lim_{x\to 0}\frac{\mathrm{arc}\,tgx}{x}=\lim_{t\to 0}\frac{t}{tgt}=\lim_{t\to 0}\frac{1}{tgt}\cdot\frac{t}{1}=\frac{1}{\lim\limits_{t\to 0}\frac{tgt}{t}}=\frac{1}{1}=1.\rhd$$

$\lhd$ **1.5** $\lim\limits_{x\to 0}\frac{shx}{x}=1;$ $\lim\limits_{x\to 0}\frac{shx}{x}=\lim\limits_{x\to 0}\frac{e^{x}-e^{-x}}{2x}=\frac{1}{2}\lim\limits_{x\to 0}\frac{e^{x}-e^{-x}}{x}=\frac{1}{2}\left(\lim\limits_{x\to 0}\frac{e^{x}-1}{x}+\lim\limits_{x\to 0}\frac{e^{-x}-1}{-x}\right)=\frac{1}{2}\cdot 2=1.$

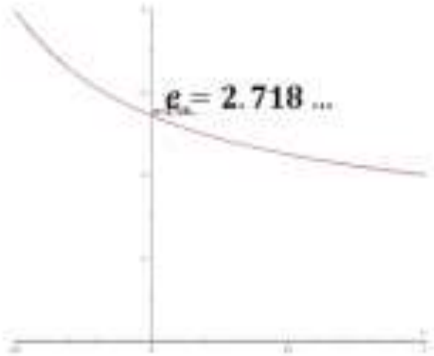
$\rhd$

$\lhd$ **1.6** $\lim\limits_{x\to 0}\frac{thx}{x}=1$

$$\lim_{x\to 0}\frac{thx}{x}=\lim_{x\to 0}\frac{\frac{shx}{chx}}{x}=\frac{1}{2}\lim_{x\to 0}\frac{e^{x}-e^{-x}}{x}\cdot\lim_{x\to 0}\frac{e^{x}+e^{-x}}{2}=\frac{1}{2}\left(\lim_{x\to 0}\frac{e^{x}-1}{x}+\lim_{x\to 0}\frac{e^{-x}-1}{-x}\right)\cdot 1=\frac{1}{2}\cdot 2=1.$$

$\rhd$

2-ajoyib limit: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \approx 2.71828\ldots$



$x_n = (1+\frac{1}{n})^n$ $n \in N$ bu ketma-ketlik e soniga teng. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ bundan biz $f(x) = \left(1+\frac{1}{x}\right)^x$ funksiyani $x \to \infty$ dagi limiti e soniga tengligini ko'rsatamiz. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ har bir $x \to +\infty$ uchun quyidagi tengsizlik o'rinli $n \le x \le n+1$

Bu yerda $n = [x] - x$ ni butun qismi

$\frac{1}{n+1} < \frac{1}{x} \le \frac{1}{n}$ bu tengsizlik $1+\frac{1}{n+1} < 1+\frac{1}{x} \le 1+\frac{1}{n}$ shuning uchun

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} \cdot \left(1+\frac{1}{n+1}\right)^{-1} = \frac{e}{1} = e$$

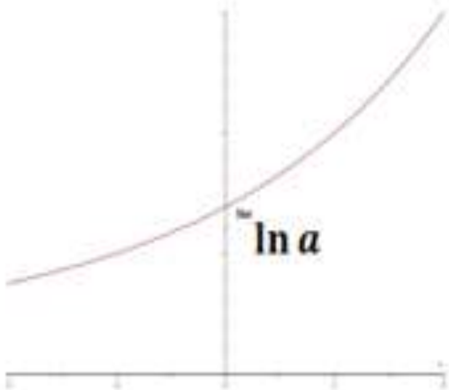
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) = e \cdot 1 = e.$$

bundan $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ kelib chiqadi.

4. $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

$$\triangleleft \lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\ln(1+x)=\lim_{x\to 0}\ln((1+x)^{\frac{1}{x}})=\ln\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\ln e=1\triangleright$$

5. $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a, \quad a>0.$



$a^x-1=y,$ deb belgilab olsak
$x=\log_a(1+y)$ $x\to 0$ da $y\to 0$

$$\lim_{y\to 0}\frac{y}{\log_a(1+y)}=\lim_{y\to 0}\frac{1}{\frac{\log_a(1+y)}{y}}=\lim_{y\to 0}\frac{1}{\log_a((1+y)^{\frac{1}{y}})}=\frac{1}{\log_a\left(\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}\right)}=\frac{1}{\log_a e}=\ln a.$$

6. $\lim_{x\to 0}\frac{e^x-1}{x}=1.$ yuqoridagidek almashtirish bajaramiz va quyidagiga ega bo'lamiz.

$$\lim_{y\to 0}\frac{y}{\ln(1+y)}=\lim_{y\to 0}\frac{1}{\frac{\ln(1+y)}{y}}=\lim_{y\to 0}\frac{1}{\ln((1+y)^{\frac{1}{y}})}=\frac{1}{\ln\left(\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}\right)}=\frac{1}{\ln e}=1.$$

7. $\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha \quad (\alpha\in\mathbb{R}).$

$(1+x)^\alpha=e^t$ deb belgilab olsak $x=e^{\frac{t}{\alpha}}-1, x\to 0\Rightarrow t\to 0$

$$\lim_{t\to 0}\frac{e^t-1}{e^{\alpha t}-1}=\lim_{t\to 0}\frac{e^t-1}{t}\cdot\frac{t}{e^{\alpha t}-1}=\lim_{t\to 0}\frac{e^t-1}{t}\cdot\frac{t}{\alpha}\cdot\frac{\alpha}{e^{\alpha t}-1}=\alpha\cdot\lim_{t\to 0}\frac{e^t-1}{t}\cdot\frac{1}{\frac{e^{\alpha t}-1}{\alpha t}}=\alpha\cdot\frac{\ln e}{\ln e}=\alpha.$$

8. $\lim\limits_{x\to 0}x^{\alpha}\ln x=\lim\limits_{x\to\infty}x^{-\alpha}\ln x=\lim\limits_{x\to\infty}x^{\alpha}e^{-x}=0,\quad (a>0).$

$\lim\limits_{x\to 0}x^{\alpha}\ln x=0,\ a>0, x^{\alpha}=t,\quad x=\sqrt[\alpha]{t}$

$$x=\sqrt[\alpha]{t}.\ln x=\ln\sqrt[\alpha]{t}=\frac{1}{\alpha}\ln t, x\to 0\quad\Rightarrow\quad t\to 0.$$

$$\lim_{t\to 0}\frac{1}{a}\ln t=\frac{1}{a}\lim_{t\to 0}\ln t=\frac{1}{a}\lim_{t\to 0}\ln(t^1)=\frac{1}{a}\lim_{t\to 0}\ln\left((1+(t-1))^t\right)=$$

$$\frac{1}{a}\lim_{t\to 0}\ln\left((1+(t-1))^{\frac{1}{t-1}}\right)=\frac{1}{a}\lim_{t\to 0}\ln\left(e^{(t-1)t}\right)=\frac{1}{a}\ln e^0=\frac{1}{a}\cdot 0=0.$$

Mavzu yuzasidan namunaviy misollar:

1-misol: $\lim\limits_{x\to 0}\frac{(1+x)^4-1-4x}{x^2}$

Yechish: Nyuton binomi formulasidan foydalanib

$(1+x)^4-1-4x=1+4x+6x^2+4x^3+x^4-1-4x=x^2(6+4x+x^2)$

$$\lim_{x\to 0}\frac{(1+x)^4-1-4x}{x^2}=\lim_{x\to 0}\frac{1+4x+6x^2+4x^3+x^4-1-4x}{x^2}=\lim_{x\to 0}\frac{x^2(6+4x+x^2)}{x^2}$$

$$\lim_{x\to 0}(6+4x+x^2)=6+0+0=6$$

2-misol: $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x-1}\right)$

Yechish: Bu limitni hisoblashda ayirmani irratsionallikdan qutqaramiz buning uchun shu ayirmani shu ayirmani yi'gindisiga ko'aytirib bo'lamiz:

$$\lim_{x\to\infty}\frac{\left(\sqrt{x^2+x+1}-\sqrt{x^2-x-1}\right)\left(\sqrt{x^2+x+1}+\sqrt{x^2-x-1}\right)}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}=$$

$$=\lim_{x\to\infty}\frac{x^2+x+1-x^2+x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}=\lim_{x\to\infty}\frac{2x+2}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}=$$

$$=\lim_{x\to\infty}\frac{2+\frac{2}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}-\frac{1}{x^2}}}=\frac{2+0}{\sqrt{1+0+0}+\sqrt{1-0-0}}=1.$$

3-misol: $\lim\limits_{x\to\frac{\pi}{6}}\frac{2\sin x-1}{x-\frac{\pi}{6}}.$

Yechish: Bu limitni hisoblashda quydagicha almashtirish bajaramiz: $x\to\frac{\pi}{6}$ bo'lgani uchun $t=x-\frac{\pi}{6}\to 0$ bo'ladi. $x=t+\frac{\pi}{6}$ berilgan limitga qo'yamiz va limitni hisoblaymiz.

$$\lim_{x\to\frac{\pi}{6}}\frac{2\sin x-1}{x-\frac{\pi}{6}}=\lim_{t\to 0}\frac{2\sin\left(t+\frac{\pi}{6}\right)-1}{t}=\lim_{t\to 0}\frac{2\left(\sin t\cos\frac{\pi}{6}+\cos t\sin\frac{\pi}{6}\right)-1}{t}=$$

$$=\lim_{t\to 0}\frac{2\left(\sin t\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\cos t\right)-1}{t}=\lim_{t\to 0}\frac{\sqrt{3}\sin t+\cos t-1}{t}=\lim_{t\to 0}\frac{\sqrt{3}\sin t+\cos t-1}{t}=$$

$$=\lim_{t\to 0}\frac{2\sqrt{3}\sin\frac{t}{2}\cos\frac{t}{2}-2\sin^2\frac{t}{2}}{t}=\lim_{t\to 0}\frac{\sin\frac{t}{2}}{\frac{t}{2}}\left(\sqrt{3}\cos\frac{t}{2}-\sin\frac{t}{2}\right)=1\cdot(\sqrt{3}-0)=\sqrt{3}.$$

4-misol: $\lim\limits_{x\to 0}\frac{3^x-\sqrt[3]{1+2x}}{\sin 5x+\ln(1-4x)}$

Yechish: Bu limitni hisoblashda funksiyaning ekvivalentidan foydalanamiz:

$$\lim_{x\to 0}\frac{3^x-\sqrt[3]{1+2x}}{\sin 5x+\ln(1-4x)}=\left[\begin{array}{l}3^x\sim 1+x\ln 3\\ \sqrt[3]{1+2x}=(1+2x)^{\frac{1}{3}}\sim 1+\dfrac{2x}{3}\\ \sin 5x\sim 5x\\ 1-4x\sim e^{-4x}\end{array}\right]=$$

$$\lim_{x\to 0}\frac{1+x\ln 3-\left(1-\dfrac{2x}{3}\right)}{5x+\ln e^{-4x}}=\lim_{x\to 0}\frac{x\left(\ln 3+\dfrac{2}{3}\right)}{x}=\ln 3+\frac{2}{3}.$$

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