UCHINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMA UCHUN NOKORREKT MASALA

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Ushbu ishda tuzilmali turdagi uchinchi tartibli xususiy hosilali differensial tenglama uchun boshlang'ich-chegaraviy masala shartli korrektlikka o'rganilgan. Masala yechimi yagonaligi va shartli turg'unligi haqidagi teoremalar isbot qilingan. Korrektlik to'plamda masalaning regulyarlashgan yechimi qurilgan.

KALIT SO'ZLAR:
tuzilmali turdagi tenglama, nokorrekt masala, yagonalik, shartli turg'unlik, regulyarlashgan yechim.

KIRISH

Ushbu ishda uchinchi tartibli tuzilmali turdagi differensial tenglama uchun boshlang'ich-chegaraviy masala shartli korrektlikka o'rganilgan.

Masalaning qo'yilishi: $\Omega = \{0 < x < \pi, 0 < t < T\}$ sohada

$$\frac{\partial}{\partial t}(u_{tt} + u_{xx}) = f(x,t) \qquad (1)$$

tenglamani va

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ u_{tt}(x,0) = h(x) \qquad (2)$$

boshlang'ich,

$$u(0,t)=u(\pi,t)=0,\ u_{xx}(0,t)=u_{xx}(\pi,t)=0 \qquad (3)$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani topish masalasini qaraymiz.

YECHIM QURISH

(1)-(3) masalani tadqiq qilish uchun

$$u_{tt}+u_{xx}=v(x,t)$$

belgilash kiritamiz. Natijada, $v(x,t)$ funksiyaga nisbatan

$$v_t=f(x,t)$$

$$v(x,0)=h(x)+\varphi''(x)$$

masalaga kelamiz. Bu yerdan

$$\int_0^t v_\tau d\tau=\int_0^t f(x,\tau)d\tau,$$

$$v(x,t)=h(x)+\varphi''(x)+\int_0^t f(x,\tau)d\tau \qquad (4)$$

bo‘lishi aniqlanadi.

Skalyar ko‘paytma $(f(x),g(x))=\int_0^\pi f(x)g(x)dx$ bo‘lsin, u holda norma $\|f(x)\|^2=\int_0^\pi f^2(x)dx$ shaklda aniqlanadi.

(4) tenglikdan

$$\|v(x,t)\|\le\|h(x)\|+\|\varphi''(x)\|+\sqrt{t}\left(\int_0^t\|f(x,\tau)\|^2d\tau\right)^{1/2} \qquad (5)$$

tengsizlik kelib chiqadi.

$u(x,t)$ funksiyaga nisbatan esa quyidagi masalaga kelamiz:

$$u_{tt}+u_{xx}=v(x,t) \qquad (6)$$

$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \qquad (7)$$

$$u(0,t)=u(\pi,t)=0 \qquad (8)$$

(6)-(8) masala yechimini $u(x,t)=\bar{u}(x,t)+\tilde{u}(x,t)$ shaklda qidiramiz, bu yerda $\bar{u}(x,t)$:

$$\bar{u}_{tt}+\bar{u}_{xx}=0$$
$$\bar{u}(x,0)=\varphi(x),\ \bar{u}_t(x,0)=\psi(x)$$
$$\bar{u}(0,t)=\bar{u}(\pi,t)=0$$

bir jinsli tenglamaga mos masalaning yechimi, $\tilde{u}(x,t)$ funksiya esa

$$\tilde{u}_{tt}+\tilde{u}_{xx}=f(x,t)$$
$$\tilde{u}(x,0)=0,\ \tilde{u}_t(x,0)=0$$
$$\tilde{u}(0,t)=\tilde{u}(\pi,t)=0$$

bir jinslimas tenglamaga mos masalaning yechimi bo'lsin.

Bir jinsli tenglamaga mos masalani yechish uchun Furye usulidan foydalanamiz, ya'ni yechimni $\bar{u}(\mathrm{x},t)=X(x)\cdot T(t)$ ko'rinishda qidiramiz va

$$X_n(x)=\sin(nx),\ T_n(t)=A_n ch(nt)+B_n sh(nt),\ \forall n\in N.$$

Bir jinsli tenglamaga mos masala yechimi quyidagi ko'rinishga ega bo'ladi:

$$\bar{u}(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n ch(nt)+\frac{\psi_n}{n}sh(nt)\right)\sin(nx)$$

bu yerda $\varphi_n=\frac{2}{\pi}\int_0^{\pi}\varphi(x)\sin(nx)dx$, $\psi_n=\frac{2}{\pi}\int_0^{\pi}\psi(x)\sin(nx)dx$.

Endi bir jinslimas tenglamaga mos masalani yechamiz va yechimni

$$\tilde{u}(x,t)=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^t sh(n(t-\tau))v_n(\tau)d\tau\sin(n\cdot x)$$

shaklda ifodalash mumkin.

Demak, (1)-(3) masala yechimi ko'rinishi quyidagicha bo'ladi:

$$u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n ch(nt)+\frac{\psi_n}{n}sh(nt)+\frac{1}{n}\int_0^t sh(n(t-\tau))v_n(\tau)d\tau\right)\sin(nx) \qquad (9)$$

bu yerda $v_n(t)=\frac{2}{\pi}\int_0^{\pi}\left(h(x)+\varphi''(x)+\int_0^t f(x,\tau)d\tau\right)\sin(nx)dx$.

YAGONALIK VA SHARTLI TURG'UNLIK TEOREMALARI

1-lemma. (1) tenglamaning $u(0,t)=u(\pi,t)=0$ shartlarni qanoatlantiruvchi yechimi uchun

$$\int_0^t\int_0^\pi u^2 dxd\tau \le d(t)\left(T\int_0^\pi u^2(x,0)dx+\gamma\right)^{1-r(t)}\left(\int_0^T\int_0^\pi u^2 dxdt+\gamma\right)^{r(t)}, \qquad (10)$$

tengsizlik o‘rinli, $t\in(0,T)$, bu yerda $d(t)=e^{\frac{2T+1}{2}\frac{\left(1-e^{-2t}\right)T-\left(1-e^{-2T}\right)t}{1-e^{-2T}}}$, $\gamma=\left(2T^2+1\right)\int_0^T\int_0^\pi v^2(x,t)dxdt+|\alpha|$, $\alpha=2\int_0^\pi\left(Tu_x^2(x,0)-Tu_t^2(x,0)+u(x,0)u_t(x,0)\right)dx$, $r(t)=\dfrac{1-e^{-2t}}{1-e^{-2T}}$.

Lemmaning isboti [8] adabiyotda keltirilgan.

(5) tengsizlikni hisobga olib, (10) tengsizlikka asosan

$$\int_0^t\int_0^\pi u^2 dxd\tau \le d(t)\left(T\int_0^\pi u^2(x,0)dx+\gamma_1\right)^{1-r(t)}\left(\int_0^T\int_0^\pi u^2 dxdt+\gamma_1\right)^{r(t)} \qquad (11)$$

tengsizlik o‘rinli bo‘ladi, bu yerda

$$\gamma_1=3\left(2T^2+1\right)\int_0^T\left(\left\|h(x)\right\|^2+\left\|\varphi''(x)\right\|^2+t\int_0^t\left\|f(x,\tau)\right\|^2 d\tau\right)dt+|\alpha|.$$

(1)-(3) masalaning korrektlik to‘plami sifatida

$$M=\left\{u(x,t):\left\|u(x,T)\right\|\le m,\ m<\infty\right\}$$

to‘plamni kiritamiz.

1-teorema. Faraz qilamiz, (1)-(3) masala yechimi mavjud va $u(x,t)\in M$ bo’lsin. U holda (1)-(3) masala yechimi yagona.

Isbot. Aytaylik, $u_1(x,t)$ va $u_2(x,t)$ funksiyalar (1)-(3) masala yechimi bo’lsin. $u(x,t)=u_1(x,t)-u_2(x,t)$ belgilash kiritamiz. U holda $u(x,t)$ funksiya

$$\frac{\partial}{\partial t}\left(u_{tt}+u_{xx}\right)=0$$

tenglamani,

$$u(x,0)=0,\ u_t(x,0)=0,\ u_{tt}(x,0)=0$$

boshlang’ich,

$$u(0,t)=u(\pi,\mathrm{t})=0,\ u_{xx}(0,t)=u_{xx}(\pi,\mathrm{t})=0$$

chegaraviy shartlarni qanoatlantiradi.

Bundan ko'rinadiki, $\gamma_1=0$ va (11) tengsizlikka asosan $u(x,t)\equiv 0$ kelib chiqadi. Demak $u_1(x,t)\equiv u_2(x,t)$, ya'ni (1)-(3) masala yechimi yagona ekan.

2-teorema. Faraz qilamiz, (1)-(3) masala yechimi mavjud va $u,u_\varepsilon\in M$ hamda $\|\varphi(x)-\varphi_\varepsilon(x)\|_{W_2^2}\le\varepsilon$, $\|h(x)-h_\varepsilon(x)\|\le\varepsilon$, $\|\psi(x)-\psi_\varepsilon(x)\|\le\varepsilon$, $\max\limits_t\|f(x,t)-f_\varepsilon(x,t)\|\le\varepsilon$ bo'lsin. U holda,

$$\int_0^t\|u-u_\varepsilon\|^2\,d\tau\le\alpha(t)\left(T\varepsilon^2+\gamma_\varepsilon\right)^{1-r(t)}\left(4Tm^2+\gamma_\varepsilon\right)^{r(t)}$$

tengsizlik o'rinli, bu yerda, $u_\varepsilon(x,t)$ - (1)-(3) masalaning $\varphi_\varepsilon(x)$, $\psi_\varepsilon(x)$, $h_\varepsilon(x)$, $f_\varepsilon(x,t)$ boshlang'ich berilganlarga mos yechimi bo'lsin, $\gamma_\varepsilon=\left(\left(2T^3+T\right)(t+2)^2+4T+2\right)\varepsilon^2$.

Isboti. $U(x,t)=u(x,t)-u_\varepsilon(x,t)$ belgilash kiritamiz. Unda $U(x,t)$ funksiya

$$\frac{\partial}{\partial t}(U_{tt}+U_{xx})=f(x,t)-f_\varepsilon(x,t)$$

tenglamani va

$$U(x,0)=\varphi(x)-\varphi_\varepsilon(x),\ U_t(x,0)=\psi(x)-\psi_\varepsilon(x),\ U_{tt}(x,0)=h(x)-h_\varepsilon(x),$$

$$U(0,t)=U(\pi,\mathrm{t})=0$$

shartlari qanoatlantiradi.

Ushbu masala uchun $U_{tt}+U_{xx}=V$ bo'lsin. U holda,

$$V_t=f(x,t)-f_\varepsilon(x,t),$$

$$V(x,0)=h(x)-h_\varepsilon(x)+\varphi''(x)-\varphi_\varepsilon''(x),$$

hamda

$$U_{tt}+U_{xx}=V$$

$$U(x,0)=\varphi(x)-\varphi_\varepsilon(x),$$

$$U_t(x,0)=\psi(x)-\psi_\varepsilon(x),$$

$$U(0,t)=U(\pi,\mathrm{t})=0$$

masalalarga kelamiz. (5) tengsizlikka ko'ra,

$$\left\|V(x,t)\right\| \le \left\|h(x)-h_\varepsilon(x)\right\| + \left\|\varphi''(x)-\varphi_\varepsilon''(x)\right\| + \sqrt{t}\cdot\left(\int_0^t \left\|f(x,\tau)-f_\varepsilon(x,\tau)\right\|^2 d\tau\right)^{\frac{1}{2}} \le \varepsilon\cdot(t+2).$$

Endi, (11) bahoga asosan

$$\alpha = 2T\int_0^\pi \left(\varphi'(x)-\varphi_\varepsilon'(x)\right)^2 dx - 2T\int_0^\pi \left(\psi(x)-\psi_\varepsilon(x)\right)^2 dx + 2\int_0^\pi \left(\varphi(x)-\varphi_\varepsilon(x)\right)\left(\psi(x)-\psi_\varepsilon(x)\right)dx$$

bundan

$$|\alpha| \le 2(2T+1)\varepsilon^2.$$

$\gamma = \left(2T^2+1\right)\int_0^T \left\|V(x,t)\right\|^2 dx + |\alpha| \le \gamma_\varepsilon$, bu yerda $\gamma_\varepsilon = \left(\left(2T^3+T\right)(t+2)^2 + 4T + 2\right)\varepsilon^2$.

Yana

$$\int_0^T \left\|U(x,t)\right\|^2 dt = \int_0^T \left\|u(x,t)-u_\varepsilon(x,t)\right\|^2 dt \le 4m^2.$$

Demak, $U = u - u_\varepsilon$ ekanligini inobatga olib, (11) tengsizlikdan

$$\int_0^t \left\|u-u_\varepsilon\right\|^2 d\tau \le \alpha(t)\left(T\varepsilon^2+\gamma_\varepsilon\right)^{1-r(t)}\left(4Tm^2+\gamma_\varepsilon\right)^{r(t)}.$$

talab qilingan tengsizlik kelib chiqadi.

REGULYARLASHGAN YECHIM QURISH

1-hol. Faraz qilamiz $\varphi(x)=0$, $\psi(x)=0$, $f(x,t)=0$, $h(x)\ne 0$ bo‘lsin. U holda $v(x,t)=h(x)$ va

$$u(x,t) = \sum_{n=1}^{\infty} \frac{h_n}{n^2}\left(ch(nt)-1\right)\sin(nx).$$

(1)-(3) masalaning regulyarlashgan yechimi sifatida

$$u_N(x,t) = \sum_{n=1}^{N} \frac{h_n}{n^2}\left(ch(nt)-1\right)\sin(nx)$$

funksiyalar ketma-ketligini olamiz, bu yerda N - regulyarlashtirish parametri. Faraz qilamiz $\left\|h(x)-h_\varepsilon(x)\right\| \le \varepsilon$ va $u(x,t)\in M$ bo‘lsin. $h_\varepsilon(x)$ taqribiy berilganga mos regulyarlashgan yechim sifatida

$$u_N^\varepsilon(x,t) = \sum_{n=1}^{N} \frac{h_{\varepsilon n}}{n^2}\left(ch(nt)-1\right)\sin(nx)$$

funksiyalar ketma-ketligini olamiz.

Ma'lumki,

$$\left\|u-u_N^\varepsilon\right\| \le \left\|u-u_N\right\| + \left\|u_N-u_N^\varepsilon\right\|.$$

Oxirgi tengsizlikning o‘ng tarafidagi ikkinchi qo‘shiluvchini baholaymiz:

$$\left\|u_N-u_N^\varepsilon\right\|^2 = \frac{\pi}{2}\sum_{n=1}^{N}\frac{\left(h_n-h_{\varepsilon n}\right)^2}{n^4}\left(ch(nt)-1\right)^2,$$

bu yerda $h_n = \frac{2}{\pi}\int_0^\pi h(x)\sin(nx)dx,\ h_{\varepsilon n} = \frac{2}{\pi}\int_0^\pi h_\varepsilon(x)\sin(nx)dx$. Bundan va $ch(nt)-1 \le sh(nt)$ tengsizlikni e'tibor olib

$$\left\|u_N-u_N^\varepsilon\right\|^2 \le \frac{\pi}{2}\sum_{n=1}^{N}\frac{\left(h_n-h_{\varepsilon n}\right)^2}{n^4}sh^2(nt) \le sh^2(Nt)\varepsilon^2$$

yoki

$$\left\|u_N-u_N^\varepsilon\right\| \le sh(Nt)\varepsilon.$$

Endi $\left\|u-u_N\right\|$ ifodani baholaymiz:

$$\left\|u-u_N\right\|^2 = \frac{\pi}{2}\sum_{n=N+1}^{\infty}\frac{h_n^2}{n^4}\left(ch(nt)-1\right)^2. \qquad (12)$$

Ma'lumki $\left\|u(x,\mathrm{T})\right\| \le m$ shartdan

$$\left\|u(x,T)\right\|^2 = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{h_n^2}{n^4}\left(ch(nT)-1\right)^2 \le m^2 \qquad (13)$$

kelib chiqadi. (13) shart ostida (12) qatorni baholash uchun shartli ekstremum Lagranj usulini qo‘llaymiz va

$$h_n = \begin{cases} \dfrac{m(N+1)^2}{ch((N+1)T)-1}\sqrt{\dfrac{2}{\pi}}, & n = N+1, \\ 0, & n \ne \mathrm{N}+1, \end{cases}$$

bo‘lishini aniqlaymiz. Natijada (12) qator uchun

$$\left\|u-u_N\right\| \le \left(\frac{ch((N+1)t)-1}{ch((N+1)T)-1}\right)m$$

baho o‘rinli. Demak

$$\left\|u-u_N^{\varepsilon}\right\|\le\left(\frac{ch\left(\left(N+1\right)t\right)-1}{ch\left(\left(N+1\right)T\right)-1}\right)m+sh\left(Nt\right)\varepsilon$$

tengsizlik o‘rinli bo‘ladi. Bundan

$$\inf_N\left\{\left(\frac{ch\left(\left(N+1\right)t\right)-1}{ch\left(\left(N+1\right)T\right)-1}\right)m+sh\left(Nt\right)\varepsilon\right\}$$

orqali regulyarlashtirish parametri N aniqlanadi.

2-hol. Faraz qilamiz $\varphi(x)=0$, $\psi(x)=0$, $h(x)=0$, $f(x,t)=f(x)$, $u(x,t)\in M$ va $\left\|f(x)-f_{\varepsilon}(x)\right\|\le\varepsilon$ bo‘lsin. U holda $v(x,t)=f(x)t$,

$$u(x,t)=\sum_{n=1}^{\infty}\frac{f_n}{n^3}\left(sh(nt)-nt\right)\sin(nx)$$

shaklda ifodalanadi. Bu holda ham 1-holga o‘xshab hisoblar bajirilganda

$$\left\|u-u_N^{\varepsilon}\right\|\le\left(\frac{sh\left((N+1)t\right)-(N+1)t}{sh\left((N+1)T\right)-(N+1)T}\right)m+\left(sh\left(Nt\right)-Nt\right)\varepsilon$$

natija kelib chiqadi.

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